Edith Cowan University

ATAR Revision Questions

ATAR Mathematics Methods

Examination Revision Questions

SOLUTIONS

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Question Number	Topic	Marks	Resource Free/Assumed
1	General Calculus	10	Free
2	General Calculus	6	Free
3	General Calculus	5	Free
4	General Calculus	6	Free
5	1^{st} and 2^{nd} derivatives	8	Free
6	1^{st} and 2^{nd} derivatives	5	Free
7	Tangents	5	Free
8	General Calculus	5	Free
9	Exponential growth	8	Assumed
10	Rectilinear motion	6	Assumed
11	Small change	7	Assumed
12	Area under a curve	5	Free
13	Total change from rate of change	5	Assumed
14	Rectilinear Motion	9	Assumed
15	Area between curves	5	Assumed
16	Binomial distribution	9	Assumed
17	Discrete random variables	9	Assumed
18	Discrete random variables	9	Assumed
19	Logarithmic Equations	11	Free
20	Rectilinear Motion	7	Free
21	Area under a curve	7	Free
22	Logarithmic functions	7	Assumed
23	Area between curves	7	Assumed
24	Uniform CRV	6	Free
25	Continuous random variables	6	Free
26	Sampling	12	Assumed
27	Normal distribution	8	Assumed
28	Continuous random variables	11	Assumed
29	Estimating areas	9	Assumed
30	Optimisation	9	Free
31	Optimisation	7	Assumed
32	Sampling	9	Free
TOTAL		238	

Question 1 (10 marks)
Determine:

(i) $\int 2x(3-x^2)^6 dx$ [2]

$$= -\int -2x\,(3-x^2)^6\,dx$$

$$= -\frac{(3-x^2)^7}{7} + C$$

(ii) $\frac{d}{dx}[\int_3^x 8-7t\,dt]$ [1]

$$= 8 - 7x$$

(iii) $\int_{-1}^{2} 4e^{-t}\,dt$ [3]

$$= \left[\frac{4e^{-t}}{-1}\right]_{-1}^{2}$$

$$= -4e^{-2} - -4e^{1}$$

$$= 4e - 4e^{-2}$$

(iv) the possible values of k, such that $\int_{\frac{\pi}{4}}^{k} \sin 2x \; dx = -0.5$ and $-\pi \leq k \leq 2\pi$. [4]

$$\left[-\frac{\cos 2x}{2} \right]_{\frac{\pi}{4}}^{K} = -0.5$$

$$-\frac{\cos 2K}{2} - \frac{-\cos \frac{\pi}{2}}{2} = -0.5$$

$$-\cos 2K + 0 = -0.5 \, (2)$$

$$\cos 2K = 1$$

$$-2\pi \leq 2k \leq 4\pi$$

$$\therefore \quad 2K = -2\pi, 0, 2\pi, 4\pi$$

$$\therefore \quad K = -\pi, 0, \pi, 2\pi$$

Question 2 (6 marks)
Determine:

(i) $\frac{d}{dx}\left(\frac{2\ln x}{3x}\right)$ [2]

$$= \frac{\frac{2}{x} \cdot 3x - 3 \cdot 2\ln x}{9x^2}$$

$$= \frac{6 - 6\ln x}{9x^2}$$

$$= \frac{2 - 2\ln x}{3x^2}$$

(ii) $\int \frac{0.5-x}{x^2-x}dx$ [2]

$$= -\frac{1}{2}\int \frac{2x-1}{x^2-x}dx \ .$$

$$= -\frac{1}{2}\ln(x^2-x) + C \quad , \text{ for } x^2-x > 0$$

(iii) $\frac{d}{dx}\left(\int_0^{\ln x} 2t\, dt\right)$ [2]

$$= (2\ln x) \cdot \frac{1}{x}$$

$$= \frac{2\ln x}{x}$$

Question 3 (5 marks)

(a) Show that $\frac{d}{dx}(xe^{3x}) = e^{3x} + 3xe^{3x}$ [1]

Product Rule: $(1)e^{3x} + 3e^{3x}(x) = e^{3x} + 3xe^{3x}$

(b) Using your answer from part (a) determine $\int xe^{3x}\,dx$. [4]

From (a) $\int e^{3x} + 3xe^{3x}\,dx = xe^{3x} + c$ [fundamental theorem of Calculus].

$$\int e^{3x}\,dx + 3\int xe^{3x}\,dx = xe^{3x} + c$$

$$3\int xe^{3x}\,dx = xe^{3x} - \int e^{3x} + c$$

$$= xe^{3x} - \frac{e^{3x}}{3} + c$$ [can leave as a single constant].

$$\int xe^{3x}\,dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c$$

Question 4 (6 marks)

Given that $\int_{-2}^{6} f(x)dx = 5$, $\int_{-2}^{9} f(x)dx = -20$ and $\int_{-2}^{10} f(x)dx = -7$, determine:

(i) $\int_{9}^{10} f(x)dx$ [2]

$$\int_{-2}^{10} f(x)\,dx - \int_{-2}^{9} f(x)\,dx = -7-(-20)$$

$$= 13$$

(ii) $\int_{6}^{10} \frac{f(x)+x}{2} dx.$ [4]

$$= \int_{6}^{10} \frac{f(x)}{2}\,dx + \int_{6}^{10} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{10} f(x)\,dx - \int_{-2}^{6} f(x)\,dx\right] + \left[\frac{x^2}{4}\right]_{6}^{10}$$

$$= \frac{1}{2}\left[-7-5\right] + \left[\frac{100}{4} - \frac{36}{4}\right]$$

$$= -6 + 25 - 9$$

$$= +10$$

Question 5 (8 marks)

(a) Determine the gradient of the curve $y = \frac{e^x}{x^3+1}$ at the point where $x = 2$. [4]

$$\frac{dy}{dx} = \frac{e^x(x^3+1) - 3x^2 e^x}{(x^3+1)^2}$$

at $x=2$

$$= \frac{e^2(9) - 12e^2}{81}$$

$$= -\frac{3e^2}{81}$$

$$= -\frac{e^2}{27}$$

(b) Determine the x coordinates of the stationary points on the function $y = x(2x-1)^3$. [4]

$$\frac{dy}{dx} = (1)(2x-1)^3 + 3(2x-1)^2(2)(x)$$

$$0 = (2x-1)^2\left[(2x-1) + 6x\right]$$

$$0 = (2x-1)^2(8x-1)$$

$\therefore$ $2x-1=0$, $x = \frac{1}{2}$ and $8x-1=0$, $x = \frac{1}{8}$ Null factor Theorem.

Question 6 (5 marks)
Consider the function $g(x) = x^3 + x - 3$.
(a) Using calculus, show that the function has zero stationary points. [2]

$$g'(x) = 3x^2 + 1$$
$$0 = 3x^2 + 1$$
$$x^2 = -\frac{1}{3}$$
$$x = \pm\sqrt{-\frac{1}{3}}$$

x has no real solutions $\therefore$ $g(x)$ has zero stationary points.

(b) Determine the coordinates of the point of inflection on $g(x)$, using calculus to prove that it is a non-stationary point of inflection. Clearly state your reasoning. [3]

$$g''(x) = 6x$$
$$0 = 6x$$
$$x = 0$$
$$g(0) = 0^3 + 0 - 3$$
$$= -3$$

$\therefore$ $(0, -3)$ is a point of inflection

as $g'(0) \neq 0$, it is a non-stationary point of inflection.

Question 7 (5 marks)
Determine the equation of the tangent to the curve $f(x) = e^x \sin(5x)$ at the point where $x = \pi$.

$$f'(x) = e^{x} \sin 5x + (5 \cos 5x) e^{x}$$

$$f'(\pi) = e^{\pi} \sin 5\pi + e^{\pi} \cdot 5 \cos 5\pi$$

$$= e^{\pi}(0) + e^{\pi} \cdot 5(-1)$$

$$= -5e^{\pi}$$

now

$$f(\pi) = e^{\pi} \sin(5\pi)$$

$$= 0$$

$\therefore$ tangent line: $y = -5e^{\pi}x + c$ with coordinate $(\pi, 0)$

$$\therefore \quad 0 = -5e^{\pi}(\pi) + c$$

$$c = 5\pi e^{\pi}$$

$\therefore$ tangent line: $y = -5e^{\pi}x + 5\pi e^{\pi}$

Question 8 (5 marks)

The function $y = ax^2 + bx^4$ has a gradient of -17 at the point $(1, \frac{-11}{2})$. Determine the values of a and b.

$$\frac{-11}{2} = a(1)^2 + b(1)^4$$

$$-11 = 2a + 2b \quad (1)$$

$$\frac{dy}{dx} = 2ax + 4bx^3$$

$$-17 = 2a(1) + 4b(1)^3$$

$$-17 = 2a + 4b \quad (2)$$

$$\therefore \quad (1) \quad -11 = 2a + 2b$$

$$(2) \quad -17 = 2a + 4b$$

$$(2) - (1)$$

$$-6 = 2b$$

$$b = -3$$

$$\therefore \quad -11 = 2a + 2(-3)$$

$$-5 = 2a$$

$$a = \frac{-5}{2}$$

Question 9 (8 marks) CA

A foreign substance is introduced into a pond. It affects the population of the fish in the pond so that the population, P varies with t according to the rule

$$\frac{dP}{dt} = -0.15P$$

where t is the number of hours since the substance was introduced.

(a) If the initial population of fish was 300 determine:

(i) an expression for P in terms of t. [1]

$$P = 300e^{-0.15t}$$

(ii) the time taken for the population to halve. [2]

$$150 = 300e^{-0.15t}$$

$$t = 4.62 \text{ hours}$$

(iii) the rate of change of the population 1 day after the substance was introduced. [2]

$$\frac{d}{dt}\left[300e^{-0.15t}\right]\Big|_{t=24} = -1.23 \text{ fish/hour}$$

(b) Determine the initial population if, after 50 hours, the population was decreasing by 0.025 fish per hour. [3]

$$P = P_0 e^{-0.15t}$$

$$\frac{dP}{dt} = -0.15P_0 e^{-0.15t}$$

$$-0.025 = -0.15P_0 e^{-0.15(50)}$$

$$P_0 \approx 301$$

Question 10 (6 marks) CA

A particle is moving in a straight line such that its displacement, x cm, at time t seconds is given by

$$x = \cos t + 0.1t^3 \quad \text{for } t \geq 0.$$

(a) Determine the displacement of the particle when $t = 3$. [1]

$x(3) = 1.71$ cm (2dp).

(b) Determine the velocity of the particle when it has an acceleration of $50cms^{-2}$ for the first time. [3]

$V = -\sin t + 0.3t^2$

$a = -\cos t + 0.6t$

$50 = -\cos t + 0.6t$

$t = 83.2826$

$V(83.2826) = 2079.80$ $\therefore$ 2079.80 m/s.

(c) At what time(s) is the speed of the particle $0.5cms^{-1}$? [2]

$|-\sin t + 0.3t^2| = 0.5$

$t = 0.71, 1.20, 2.12$

Question 11 (7 marks) CA

(a) Given that $y = \sqrt{x} + x$, use calculus to approximate the small change in y when x changes from 5 to 5.01. [2]

$$\frac{dy}{dx}\bigg|_{x=5} = 1.224$$

$$\delta y = 1.224\,(0.01)$$

$$\sim 0.012$$

(b) The incremental formula, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$, is used to approximate the small change in the function $y = e^{ax} + bx$. It approximates that the small change in y when x changes from 2 to 2.05 is 1.5 and the small change in y when x changes from 10 to 10.05 is 3.5. Determine the values of a and b, correct to 3 significant figures. [5]

$$\frac{dy}{dx} = ae^{ax} + b$$

$$\delta y = (ae^{ax} + b)\,\delta x$$

$$\left\{\begin{array}{l} 1.5 = [ae^{2a} + b][0.05] \\ 3.5 = [ae^{10a} + b][0.05] \end{array}\right. \Bigg|\ a, b$$

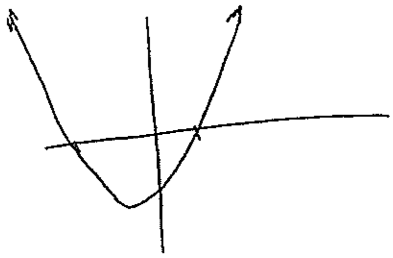
$$a = 0.451,\ b = 28.9$$

Question 12 (5 marks)

Determine the area between the curve $y = x^2 + 3x - 4$ and the x axis between $x = 0$ and $x = 2$.

Determine the roots

$$0 = x^2 + 3x - 4$$



$$0 = (x+4)(x-1)$$

$$x = -4, 1$$

$$\therefore -\int_0^1 x^2 + 3x - 4 \, dx + \int_1^2 x^2 + 3x - 4 \, dx$$

$$= -\left[\frac{x^3}{3} + \frac{3x^2}{2} - 4x\right]_0^1 + \left[\frac{x^3}{3} + \frac{3x^2}{2} - 4x\right]_1^2$$

$$= -\left(\frac{1}{3} + \frac{3}{2} - 4\right) + \left(\left(\frac{8}{3} + 6 - 8\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)\right)$$

$$= -2\left(\frac{1}{3} + \frac{3}{2} - 4\right) + \frac{8}{3} - 2$$

$$= -\frac{2}{3} - \frac{6}{2} + 8 + \frac{8}{3} - 2$$

$$= \frac{6}{3} + 3$$

$$= 5 \text{ units}^2$$

Question 13 (5 marks) CA

The volume of a balloon, $V\ cm^3$, is varying such that after t minutes, $\frac{dV}{dt} = \frac{3}{\sqrt{t}}$.

(a) Determine, correct to 5 significant figures, the change in volume from $t = 10$ to $t = 16$. [2]

$$\int_{10}^{16} \frac{3}{\sqrt{t}}\, dt = 5.0263\ cm^3$$

(b) Determine the time time taken, to the nearest second, from when $t = 50$ for the volume to increase by $40\ cm^3$. [3]

$$\int_{50}^{\cancel{5}0+x} \frac{3}{\sqrt{t}}\, dt = 40$$

$$t = 138.7253$$

138 mins, 44 seconds.

Question 14 (9 marks) CA

A particle, with an initial displacement of -0.25 meters, is moving in a straight line such that at time t seconds, its velocity is given by $v(t) = 4\sin(\frac{t}{6})$ meters per second, where $t \geq 0$.

(a) Determine the average speed of the body in the first minute. [2]

$$= \frac{\int_0^{60} \left|4 \sin \frac{t}{6}\right| dt}{60 - 0}$$

$$= 2.46 \text{ m/s}$$

(b) Determine the smallest value of q, such that the average velocity from $t = 1$ to $t = q$ is $1.75\, m/s$. [3]

$$\frac{\int_1^{q} \sin \frac{t}{6}\, dt}{q - 1} = 1.75 \quad , \quad q = 4.52$$

(c) Determine the acceleration of the particle when it is at the origin for the first time. [4]

$$x(t) = -24 \cos \frac{t}{6} + c$$

$$-0.25 = -24 \cos 0 + c \, , \quad c = 23.75$$

$$\therefore \quad 0 = -24 \cos \frac{t}{6} + 23.75$$

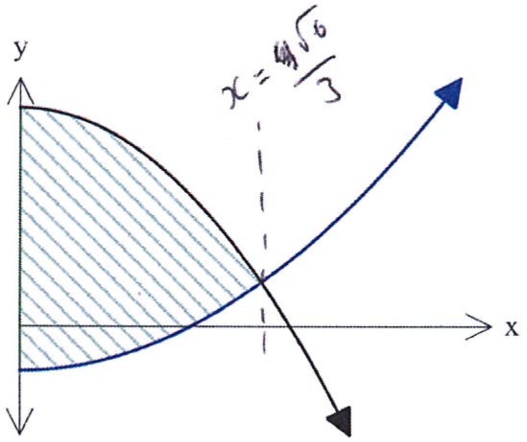
$$t = 0.86678$$

$$\therefore \quad x''(0.86678) = 0.660 \quad [\text{from CAS}]$$

$$\therefore \quad 0.660 \text{ m/s/s.}$$

Question 15 (5 marks) CA

The graph below shows the functions $f(x) = 3x^2 - 1$ and $g(x) = ax^2 + b$, intersecting at $x = \frac{\sqrt{6}}{3}$. Given that the shaded area is equal to $\frac{4\sqrt{6}}{3}$ units squared, determine the values of a and b.



$$f\left(\frac{\cancel{4}\sqrt{6}}{3}\right) = g\left(\frac{\cancel{4}\sqrt{6}}{3}\right) \quad (1)$$

$$\int_0^{\cancel{4}\sqrt{6}/3} g(x) - f(x)\, dx = \frac{4\sqrt{6}}{3}$$

$$\begin{cases} 3\left(\frac{\sqrt{6}}{3}\right)^2 - 1 = a\left(\frac{\sqrt{6}}{3}\right)^2 + b \\ \int_0^{\sqrt{6}/3} ax^2 + b - (3x^2 - 1)\, dx = \frac{4\sqrt{6}}{3} \end{cases}, a, b$$

$$a = -6, \quad b = 5.$$

Question 16 (9 marks) CA

When throwing a dart at a dart board, Taylor has a 0.18 chance of hitting the bulls eye.

(a) If Taylor has 30 throws at the dart board determine the probability that he hits:

(i) exactly 2 bulls eyes. [2]

if X = Nº bulls eyes , $X \sim Bin[30, 0.18]$

$P(X=2) = 0.05443$

(ii) more than 3 bulls eyes. [1]

$P(X \geq 4) = 0.81436$

(iii) no more than 6 bulls eyes, given he hits more than 3 bulls eyes. [2]

$$\frac{P(4 \leq X \leq 6)}{P(X \geq 4)} = 0.64630$$

(b) Taylor plays 5 games of darts, each time throwing 30 darts. Determine the probability he hits exactly 2 bulls eyes in at least 3 of the 5 games. [2]

let Y = Nº times he hits exactly two bulls eyes

$Y \sim Bin[5, 0.05443]$

$P(Y \geq 3) = 0.0015$

(c) How many darts would Taylor need to throw so that he has at least a 75% chance of hitting at least one bulls eye? [2]

if $P(X \geq 1) \geq 0.75$ then $P(X=0) < 0.25$

$\therefore (1-0.18)^n < 0.25$

$n > 6.99 \therefore 7$ throws.

Question 17 (9 marks) CA
Consider the probability function shown below.

x	1	3	m	8
$P(X = x)$	0.15	n	0.3	0.35

(a) Determine n. [1]

$n = 0{,}2$

(b) If the expected value of X is 5.65, determine:

(i) m. [2]

$1(0.15) + 3(0.2) + m(0.3) + 8(0.35) = 5.65$

$m = 7$.

(ii) $Var(X)$. [2]

From CAS $\quad$ ST DEV (x) = 2.669738^2

$= 7.1275$

(iii) $E(3 - 1.5X)$. [1]

$= 3 - 1.5(5.65)$

$= -5.475$

(iv) a and b such that $E(a + bX) = 12$ and $Var(a + bx) = 26$ and $b > 0$.

[3]

$7.1275(b^2) = 26$

$b = \cancel{-1.91}, 1.91$

reject -1.91 as $b > 0$.

$a + (1.9099)(5.65) = 12$

$a = 1.21$

Question 18 (9 marks) CA

At the Royal Show, a game stall operator offers prizes of \$1, \$3, and \$10 with probabilities of 0.45, 0.2 and 0.05 respectively. The operator charges \$2 per game.

(a) Show that the probability of not winning a prize is 0.3. [1]

$$1 - 0.45 - 0.2 - 0.05 = 0.3$$

(b) Determine the probability of a player winning \$3, given that that win a prize. [2]

$$\frac{0.2}{0.7} = \frac{2}{7}$$

(c) After 200 plays of the game, how much should the game stall operator expect to profit? [2]

Profit	2	1	−1	−8
	0.3	0.45	0.2	0.05

From CAS, $E(\text{Profit}) = 0.45$

$$200 \times 0.45 = \$90$$

(d) If Sydney plays the game 35 times, what is the probability she gets a prize of exactly \$3 more than 5 times? [2]

let Y = Nº times Sydney wins \$3

$Y \sim Bin[35, 0.2]$

$P(Y \geq 6) = 0.72791$

(e) How much should the game stall operator charge if she wishes to make \$150 profit in 200 plays of the game? [2]

$\frac{150}{200} = 0.75$ i.e 75¢ per game.

$\therefore$ needs an extra 30¢. (as E(Profit) = 45¢).

$\therefore$ \$2.30 / game.

Question 19 (11 marks)

(a) Determine the exact value of x in each of the following equations.

(i) $3^x = 2^{5-x}$ [3]

$x \ln 3 = (5 - x) \ln 2$

$x \ln 3 = 5 \ln 2 - x \ln 2$

$x \ln 3 + x \ln 2 = 5 \ln 2$

$x = \dfrac{5 \ln 2}{\ln 3 + \ln 2}$

(ii) $\ln(\tan x) + 0.5 \ln 3 = 0$ for $0 < x < \frac{\pi}{2}$. [2]

$\ln(\tan x) = -\ln \sqrt{3}$

$\ln \tan x = \ln\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \tan x = \dfrac{1}{\sqrt{3}}$

$x = \dfrac{\pi}{6}$

(b) If $\log_2 9 = a$ and $\log_2 6 = b$ express the following in terms of a and b.

(i) $-\log_2 54$. [2]

$= -(\log_2 9 + \log_2 6)$

$= -(a + b)$

(ii) $\log_2 36$. [1]

$= 2 \log_2 6$

$= 2b$

(iii) $\log_2 0.75$ [3]

$= \log_2 \frac{3}{4}$

$= \log_2 \frac{6}{8}$

$= \log_2 6 - \log_2 8$

$= b - 3$

* other answers exist.

Question 20 (7 marks)

A particle moves in a straight line such that its displacement, x metres, at time t seconds is given by

$$x(t) = -1 + \ln(t^2 - 2t + 2) \text{ for } t \geq 0.$$

(a) Determine when the particle is stationary and its distance from the origin at this time. [4]

$$v(t) = \frac{(2t-2)}{t^2-2t+2}$$

$$\therefore 0 = 2t - 2$$

$$t = 1$$

$$\therefore x(1) = -1 + \ln(1)$$

$$= -1$$

$\therefore$ Distance is 1m from origin.

(b) Determine the acceleration of the particle when $t = 3$. [3]

$$a(t) = \frac{(t^2-2t+2)(2)-(2t-2)(2t-2)}{(t^2-2t+2)^2}$$

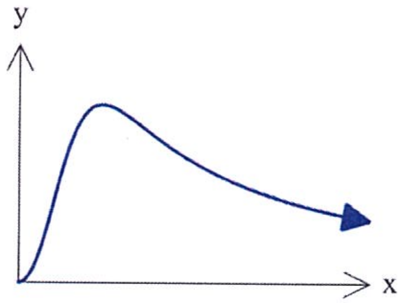
$$= \frac{2(9-6+2)-(4)(4)}{(9-6+2)^2}$$

$$= \frac{10-16}{25}$$

$$= -\frac{6}{25} \text{ ms}^{-2}$$

Question 21 (7 marks)

Consider $f(x) = \frac{3x^2}{x^3+1}$ graphed below.



(a) Determine the area bound by $f(x)$ and the x-axis from $x = 1$ to $x = 3$, expressing your answer as a single logarithm. [3]

$$\int_1^3 \frac{3x^2}{x^3+1}\,dx = \left[\ln(x^3+1)\right]_1^3$$

$$= \ln 28 - \ln 2$$

$$= \ln 14 \text{ units}^2.$$

(b) The area bound by $f(x)$ and the x-axis from $x = 1$ to $x = k$ is $\ln 63$ units squared. Determine the value of k. [4]

$$\left[\ln(x^3+1)\right]_1^k = \ln 63$$

$$\ln(k^3+1) - \ln 2 = \ln 63$$

$$\ln(k^3+1) = \ln(126)$$

$$k^3 = 125$$

$$k = 5$$

Question 22 (7 marks) CA

The loudness of a sound, L decibels, is related to the intensity of the sound, I by the equation

$$L = 10\log\frac{I}{I_0}$$

where I_0 is a constant.

(a) Determine the loudness of a sound when its intensity is $150I_0$. [2]

$L = 10 \log 150$

$= 21.76$ decibels.

(b) How many times more intense is a sound with a loudness of 75 decibels to that of a sound with a loudness of 48 decibels, correct to 3 significant figures. [2]

$I = I_0 \cdot 10^{I/10}$

$\therefore \dfrac{I_0 \cdot 10^{75/10}}{I_0 \cdot 10^{48/10}} = 10^{2.7}$

~ 501

$\left(L = 10 \log \frac{I}{I_0}\right)$

solve for I in CAS.

(c) Use calculus to approximate the small change in L when I increases to 1.1×10^{-4} from 10^{-4}. [3]

$\dfrac{d}{dI}\left[10 \log \frac{I}{I_0}\right]\bigg|_{I=10^{-4}} = 43429.45$

$\delta I = 10^{-4} - 1.1\times10^{-4}$

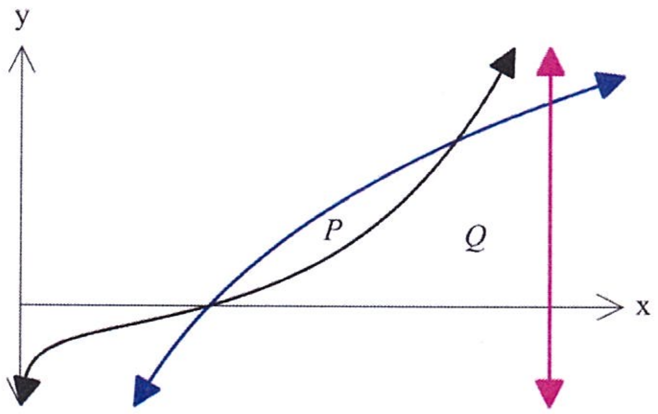
$= 0.00001$

$\therefore \delta L \approx 43429.45 \times 0.00001$

~ 0.4343 decibels

Question 23 (7 marks) CA

The functions $f(x) = 5\ln x$ and $g(x) = \frac{e^x \ln x}{2}$ and the line $x = b$ are graphed on the axes below. P is a region bounded by $f(x)$ and $g(x)$ and Q is the region bounded by the two curves, the line $x = b$ and the x-axis.



(a) Determine an integral that will give the exact area of region P, and state this area correct to 3 significant figures. [4]

$$5\ln x = \frac{e^x \ln x}{2} \quad \therefore \text{Area}(P) = \int_1^{\ln 10} 5\ln x - \frac{e^x \ln x}{2}\, dx$$

$$x = 1, \ln 10. \qquad = 1.05 \text{ units}^2.$$

(b) Region Q is 5 units squared. Determine the value of b correct to 3 significant figures. [3]

$$\int_1^{\ln 10} \frac{e^x \ln x}{2}\, dx + \int_{\ln 10}^{b} 5\ln x\, dx = 5$$

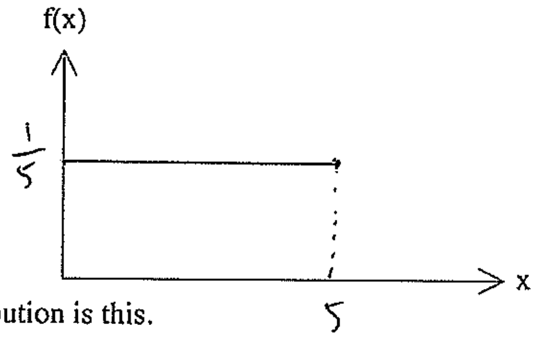
$$b = 2.92$$

Question 24 (6 marks)

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0.2 & for\ 0 \le x \le 5 \\ 0 & elsewhere. \end{cases}$$

(a) Graph this distribution on the axes below, showing clearly the scale on each axis. [1]



(b) What type of distribution is this. [1]

Uniform.

(c) Determine:

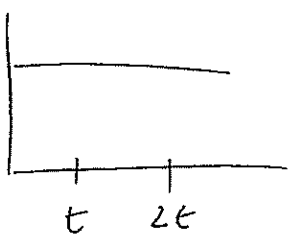
(i) $P(X \le 4)$. [1]

$\frac{4}{5}$

(ii) $P(X \le 4 | X \ge 3)$. [1]

$\frac{1}{2}$

(iii) t such that $P(t < X < 2t) = \frac{1}{3}$. [2]



$\therefore\ (2t - t) \times \frac{1}{5} = \frac{1}{3}$

$t = \frac{5}{3}$

Question 25 (5 marks)

The Probability density function for the continuous random variable X is given by

$$f(x) = \begin{cases} mx + c, & 0 \leq x \leq 3 \\ 0 \; elsewhere \end{cases}$$

Given than $E(X) = 1$, determine the values of m and c.

$$\int_0^3 mx + c \, dx = 1 \quad ✓$$

$$\left[\frac{mx^2}{2} + cx\right]_0^3 = 1$$

$$(2) \quad \frac{9m}{2} + 3c = 1$$

$$\int_0^3 x(mx+c)\,dx = 1 \quad ✓$$

$$\int_0^3 mx^2 + cx \, dx = 1$$

$$\left[\frac{mx^3}{3} + \frac{cx^2}{2}\right]_0^3 = 1 \quad ✓$$

$\times 2$

$$(1) \quad 9m + \frac{9c}{2} = 1$$

$$2 \times (2) \quad 9m + 6c = 2$$

$$\frac{-3c}{2} = -1$$

$$c = \frac{2}{3} \quad ✓$$

$$\therefore \; 9m + 6 \cdot \frac{2}{3} = 2$$

$$9m = -2$$

$$m = \frac{-2}{9} \quad ✓$$

5

Question 26 (12 marks) CA

A random sample of 400 fish from a large pond is caught. 15 of these fish have a defect with their tail.

(a) Determine a point estimate for the proportion of fish in the entire pond that have a defect with their tail. [1]

$$\hat{p} = \frac{15}{400}$$

$$= 0.0375$$

(b) Determine a 95% confidence interval for the proportion of fish in the pond that have a defect with their tail. [3]

Margin of error: $E = 1.960\sqrt{\frac{0.0375(1-0.0375)}{400}}$

$$= 0.0186$$

$\therefore \quad 0.0375 \pm 0.0186$

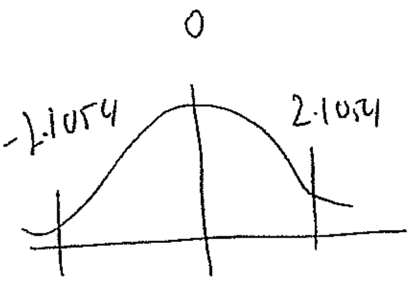
or $\quad (0.0189, 0.0561)$

(c) From the sample taken, Andrea is $x\%$ confident that the proportion of fish in the pond with a tail defect lies between 0.0175 and 0.0575. Determine the value of x correct to one decimal place. [5]

$$E = \frac{0.0575 - 0.0175}{2} = 0.02$$

$$\therefore K\sqrt{\frac{0.0375(1-0.0375)}{400}} = 0.02$$

$$K = 2.1054$$



$Z \sim N[0,1]$

$$P(-2.1054 < Z < 2.1054) = 0.965$$

$$\therefore 96.5 = x.$$

(d) Seventy samples are taken and the 97% confidence intervals for the proportion of fish in the pond with a tail defect, p, are taken. If $X =$ the number of samples who confidence interval contains the true value of p, state the distribution of X, stating its parameters and mean and standard deviation. [3]

$X \sim Bin[70, 0.97]$

$$\text{mean} = 70(0.97)$$
$$= 67.9$$

$$SD = \sqrt{[70(0.97)(1-0.97)]}$$
$$= 1.427$$

Question 27 (8 marks) CA

A tackle company sells sinkers. The weight of the sinkers is normally distributed with a mean of $80g$ and a standard deviation of $2g$.

A sinker is selected at random. $X \sim N[80, 2^2]$

(a) Determine the probability that the sinker weighs:

(i) more than $78g$, [1]

$P(X > 78) = 0.84134$

(ii) less than $81g$, given that it weighs more than $78g$. [2]

$$\frac{P(78 < X < 81)}{P(X > 78)} = 0.6333$$

(b) If 70 sinkers were selected at random, determine the probability that exactly 50 of them weigh more than $78g$. [2]

let Y = Nº sinkers above 78g

$\therefore\ Y \sim Bin[70, 0.84134]$

$P(Y=50) = 0.0029$

The machine that produces the sinkers is set to $80g$. What ever amount the machine is set to will be the mean of the normal distribution, with the standard deviation always remaining at $2g$.

(c) What should the machine be set to, if the company wants 98.5% of the sinkers produced to be more than $80g$? [3]

using classpad (CAS).

$$\text{Norm CDF}(80, \infty, 2, x) = 0.985$$

$$x = 84.34$$

Question 28 (11 marks) CA

The continuous random variable, X, has probability density function

$$f(x) = \begin{cases} \frac{\sqrt{3x}}{6}, & for\ \ 0 \le x \le 3 \\ 0\ for\ all\ other\ values\ of\ x. \end{cases}$$

(a) Determine $P(1 < X < 2)$. [1]

$$\int_1^2 \frac{\sqrt{3x}}{6}\,dx = 0.35188$$

(b) Determine q such that $P(0 < X < q) = 0.5$ [2]

$$\int_0^q \frac{\sqrt{3x}}{6}\,dx = 0.5$$

$$q = 1.88989$$

(c) Determine:

(i) $E(X)$ [2]

$$\int_0^3 x\cdot\frac{\sqrt{3x}}{6}\,dx = 1.8$$

(ii) $Var(X)$ [2]

$$\int_0^3 (1.8 - x)^2\cdot\frac{\sqrt{3x}}{6}\,dx = 0.61714$$

(iii) $E(5X - 3)$ [1]

$$= 5(1.8) - 3$$

$$= 6$$

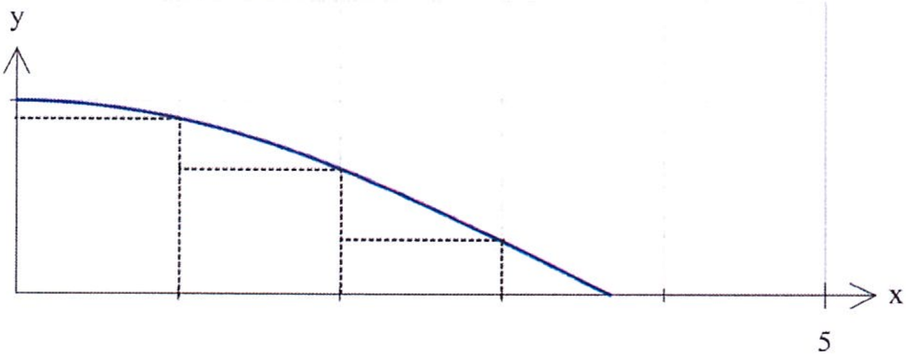
(d) Determine the cumulative distribution function for X. [3]

$$\int_0^x \frac{\sqrt{3t}}{6}\,dt = \frac{\sqrt{3}\,x^{1.5}}{9}$$

$$\therefore \quad P(X \le x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{\sqrt{3}\,x^{1.5}}{9} & \text{for } 0 \le x \le 3 \\ 1 & \text{for } x > 3 \end{cases}$$

Question 29 (9 marks) CA

The function $f(x) = \cos\frac{3x}{7}$ is graphed below.



(a) Complete the table below. [1]

x	0	1	2	3
$f(x)$	1	0.9096	0.6546	0.2812

(b) Use the inscribed rectangles of width 1 unit, shown on the axes above, to underestimate the area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$. [2]

$= 1 \times [0.9096 + 0.6546 + 0.2812]$

$= 1.8454$

(c) Use circumscribed rectangles of width 1unit to overestimate the area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$. [2]

$= 1 \times [1 + 0.9096 + 0.6546]$

$= 2.5642$

(d) By averaging out your results from (b) and (c), determine a better estimate for the area between $f(x)$ and the x-axis from $x = 0$ to $x = 3$. [1]

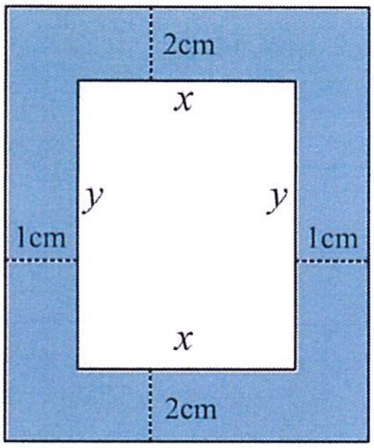
$= 2.2048$

(e) Using calculus, determine the exact area of $f(x)$ and the x axis from 0 to 3. [3]

$$\int_0^3 \cos\left[\frac{3x}{7}\right] dx = 2.239.$$

Question 30 (11 marks)

A picture frame (shaded below) is being designed with a $2cm$ border at the top and bottom and a $1cm$ border at the sides, with the area for the picture being fixed at $200cm^2$.



(a) Show that the area of the picture frame, A, can be modelled by the equation

$$A = \frac{400}{x} + 4x + 8$$ [3]

$A = (x+2)(y+4) - xy$

now, $xy = 200$

$\therefore y = \frac{200}{x}$

$A = (x+2)\left(\frac{200}{x} + 4\right) - 200$

$A = 200 + 4x + \frac{400}{x} + 8 - 200$

$A = \frac{400}{x} + 4x + 8$

(b) Determine the values of x and y that will minimise A, using $\frac{d^2A}{dx^2}$ to prove it is a minimum. [5]

$$\frac{dA}{dx} = \frac{-400}{x^2} + 4$$

$$0 = \frac{-400}{x^2} + 4$$

$$-4x^2 = -400$$

$$x^2 = 100$$

$$x = \cancel{-10}, 10$$

$$\therefore y = \frac{200}{10}$$

$$y = 20$$

$$\frac{d^2A}{dx^2} = \frac{800}{x^3}$$

$$= \frac{800}{20^3}$$

which is positive

$\therefore$ at $x = 20$, A is a minimum TP.

(c) Using the incremental formula, approximate the change in A, when x changes from $2cm$ to $1.99cm$. [3]

at $x = 2$, $\frac{dA}{dx} = \frac{-400}{2^2}$

$$= -100 + 4 = -96$$

$$\therefore \delta A \approx -96(-0.01)$$

$$\approx -0.96$$

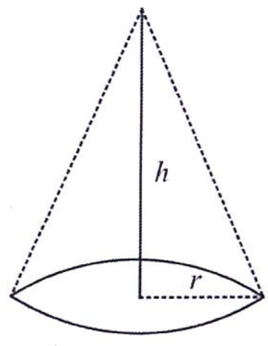
$\therefore$ $0.96\ cm^2$

Question 31 (7 marks) CA

A tent in the shape of a cone is to be pitched. A bamboo frame is needed for the circumference of the base and the height of the cone. 8 metres of bamboo to be used for the framework, represented by the the solid lines in the diagram below.

(a) Show that the volume V, of the tent in terms of its radius r, is given by [2]

$$V = \frac{8}{3}\pi r^2 - \frac{2}{3}\pi^2 r^3$$



$V = \frac{\pi r^2 h}{3}$

now, $2\pi r + h = 8$

$\therefore h = 8 - 2\pi r$

$\therefore V = \frac{\pi (8 - 2\pi r)^2 h}{3}$

$V = \frac{8\pi r^2}{3} - \frac{2\pi^2 r^3}{3}$

(b) Determine the radius of the tent that will maximise the volume, leaving your answer in terms of π. State this maximum volume and prove it is indeed a maximum. [5]

$\frac{dV}{dr} = -(6r^2\pi^2 - 16x\pi)/3$

$0 = -(6r^2\pi^2 - 16x\pi)/3$

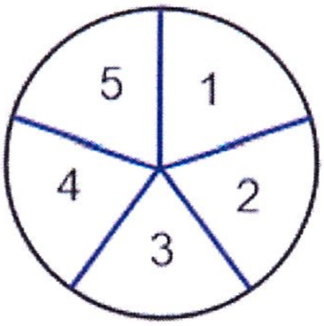
$r = \frac{8}{3\pi}$ or 0.8488

$\frac{d^2V}{dr^2}$ at $r = \frac{8}{3\pi}$ is $-\frac{16\pi}{3}$

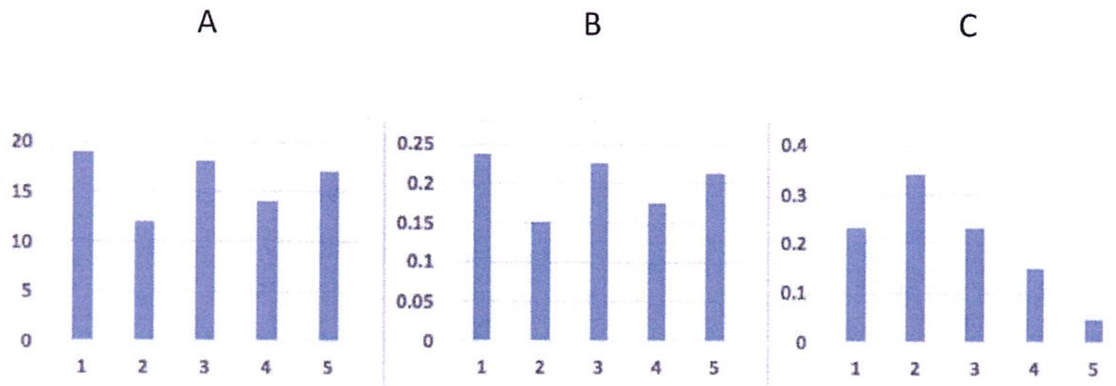
$\therefore$ V is concave down, here, a maximum turning point

Question 32 (9 marks) CA

The spinner pictured below is such that each outcome $(1, 2, 3, 4$ and $5)$ is equally likely. An experiment consists of spinning the spinner 80 times.



(a) In the experiment, the frequencies of each outcome were noted. Which histogram below is most likely to show this data, if the outcome is listed on the x-axis and the relative frequency is listed on the y-axis? Explain your answer. [2]



B. The distribution is reasonably uniform. The sum of the frequencies add to one, hence, they are relative frequencies

(b) If X is the number of times that an odd number is spun:

(i) describe the distribution of X, stating its parameters. [2]

$$X \sim Bin\left[80, 0.6\right]$$

(ii) determine the probability that X is greater than 40. [1]

$$P(X \geq 41) = 0.95550$$

(c) The experiment involving 80 spins of the spinner is repeated 760 times. In how many of the 760 experiments, would you expect the proportion of odd numbers to be between 0.35 and 0.65 ? [4]

$$\hat{p} \sim N\left[0.6, \sqrt{\frac{0.6(1-0.6)}{80}}^{2}\right]$$

$$P(0.35 < \hat{p} < 0.65) = 0.8193$$

$$0.8193 \times 760 \approx 623$$

$$\therefore 623$$