Edith Cowan University

ATAR Revision Questions

ATAR Mathematics Specialist

Examination Revision Questions

SOLUTIONS

Prepared and presented by

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Question Number	**Topic**	**Marks**	**Resource Free/Assumed**
1	Complex Numbers	10	Free
2	Complex Numbers	5	Free
3	Complex Numbers	5	Free
4	Complex numbers	4	Free
5	Functions	4	Free
6	Functions	7	Free
7	Vectors Calculus	7	Free
8	Lines	6	Free
9	Lines/Spheres	7	Free
10	Systems of Equations	5	Free
11	Plane	5	**Assumed**
12	Vectors in Geometry	7	**Assumed**
13	Implicit Differentiation	9	Free
14	Integration	9	Free
15	Integration	4	Free
16	Applications of Differentiation	11	**Assumed**
17	Applications of Differentiation	6	**Assumed**
18	Parametric Equations	5	**Assumed**
19	SHM	5	**Assumed**
20	Slope Fields	6	Free
21	Logistic Equation	6	**Assumed**
22	SHM	7	**Assumed**
23	Sampling	4	**Assumed**
24	Sampling	4	**Assumed**
25	Separation of Variables	5	**Assumed**
26	Rectilinear Motion	7	**Assumed**
27	Complex Numbers.	10	Free
28	Sampling	8	**Assumed**

Question 1 (10 marks)

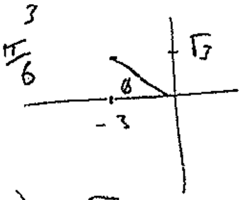
Given $z = -3 + \sqrt{3}\, i$, express z in the form $r\ cis\ \theta$ $(r > 0, -\pi < \theta \leq \pi)$:

(i) z [2]

$Z = 2\sqrt{3}\ cis\ \frac{5\pi}{6}$ ✓✓

$\tan \phi = \frac{\sqrt{3}}{3}$

$\phi = \frac{\pi}{6}$



$r = \sqrt{(9+3)} = \sqrt{12} = 2\sqrt{3}$

(ii) $5z$ [1]

$5z = 10\sqrt{3}\ cis\ \frac{5\pi}{6}$ ✓

(iii) z^2 [2]

$Z^2 = 12\ cis\ \frac{10\pi}{6}$ ✓

$= 12\ cis\ \frac{5\pi}{3}$

$= 12\ cis\ \frac{-\pi}{3}$ ✓

(iv) $\bar{z}$ [1]

$\bar{Z} = 2\sqrt{3}\ cis\ -\frac{5\pi}{6}$ ✓

(v) $\frac{1}{z} = Z^{-1}$ [2]

$= \frac{1}{2\sqrt{3}} \operatorname{cis} \frac{-5\pi}{6}$ ✓✓

$= \frac{\sqrt{3}}{6} \operatorname{cis} -\frac{5\pi}{6}$

(vi) $z + 6$ [2]

$Z + 6 = -3 + \sqrt{3}i + 6$

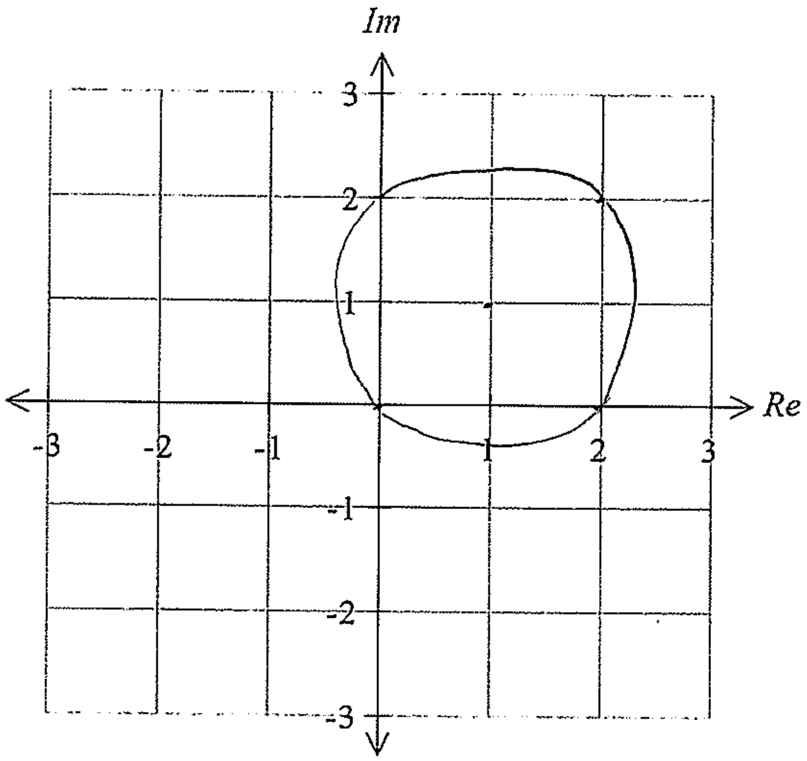
$= 3 + \sqrt{3}i$ ✓

$= 2\sqrt{3} \operatorname{cis} \frac{\pi}{6}$ ✓

Question 2 (5 marks)

(i) Sketch, on the Argand diagram below, the set of points given by

$\{z: |z - i - 1|^2 = 2\}$. [3]

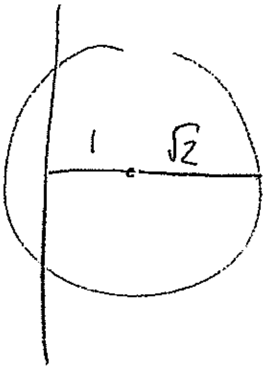


$$\sqrt{(x-1)^2 + (y-1)^2}^{\,2} = 2$$

$$(x-1)^2 + (y-1)^2 = 2$$

i.e circle with centre at $(1,1)$
and radius $\sqrt{2}$.

(ii) Determine the maximum value of $Re(z)$ for the values of z in the set defined in part (i). [2]



$1 + \sqrt{2}$

Question 3 (5 marks)

(i) Use de Moivre's theorem to find all three solutions of the equation $z^3 = 8$, expressing your answers in rectangular form. [3]

$z^3 = 8 \text{ cis } 0$ ✓

$z = 2 \text{ cis } 0$ (PR) ✓

$2\pi/3$ apart

$\therefore z = 2\text{cis } 0, \; 2\text{cis}\frac{2\pi}{3}, \; 2 \text{ cis}\left(-\frac{2\pi}{3}\right)$

$= 2, \; 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \; 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 2, \; 1 + \sqrt{3}i, \; 1 - \sqrt{3}i$ ✓

(ii) Hence write down all three solutions of the equation $(z+1)^3 = 8$ in rectangular form. [2]

$z + 1 = 2, \; 1 + \sqrt{3}i, \; 1 - \sqrt{3}i$ ✓

$z = 1, \; \sqrt{3}i, \; -\sqrt{3}i$ ✓

5

Question 4 (4 marks)

Given that $z = \sqrt{5}\,i$ is a solution of the equation

$$2z^3 - 3z^2 + 10z = 15 \ ,$$

find the other two solutions.

$-\sqrt{5}i$ is a solution ✓

$\therefore$ $(z - \sqrt{5}i)(z + \sqrt{5}i) = z^2 + 5$ is a factor

$$\begin{array}{r|l} & \quad 2z \quad - 3 \\ \hline z^2 + 5 & 2z^3 - 3z^2 + 10z - 15 \\ & 2z^3 \qquad\quad + 10z \\ \hline & 0 \quad - 3z^2 + 0 \\ & \qquad - 3z^2 + 0 \quad - 15 \\ \hline & \qquad\qquad\qquad\quad 0 \end{array}$$ ✓

$\therefore$ $2z - 3$ is a factor

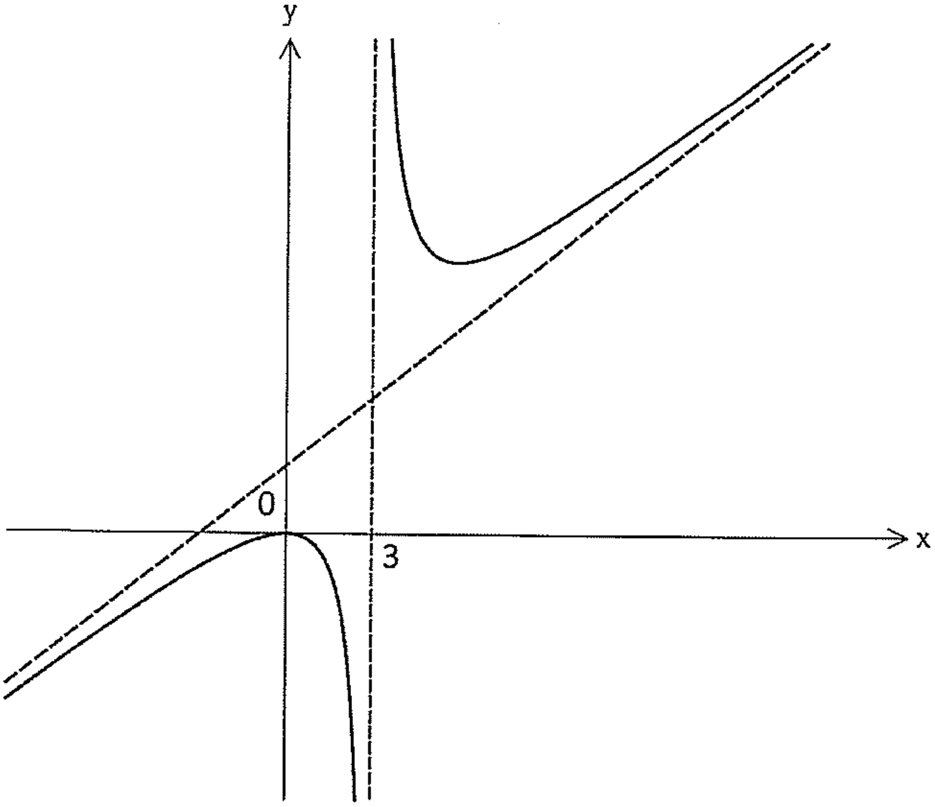
$\therefore$ $2z - 3 = 0$ ✓

$z = \frac{3}{2}$

thus $z = \sqrt{5}i \ , \ -\sqrt{5}i \ , \ \frac{3}{2}$ ✓

Question 5 (4 marks)

The graph of $y = \frac{x^2}{x-k}$ $(k > 0)$ is shown below:



(i) State the value of k. [1]

3

(ii) Determine the equation of the inclined asymptote. [3]

$x + 3$

$x - 3 \overline{) \, x^2}$

$x^2 - 3x$

$3x$

$3x - 9$

9

$y = x + 3 + \frac{9}{x-3}$

Asymptote: $y = x + 3$

Question 6 (7 marks)

Given $f(x) = \sqrt{x-1}$

and $g(x) = |x|$,

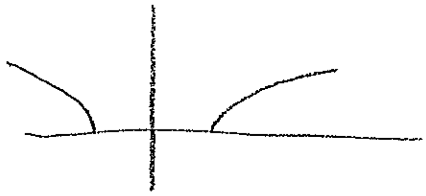
(i) determine $(f \circ g)(x)$ and state its domain and range, [4]

$f \circ g(x) = \sqrt{|x| - 1}$ ✓

$\mathcal{D} : \{x \in \mathbb{R} : x \leq -1 , x \geq 1\}$ ✓✓

$\mathcal{R} : \{y \in \mathbb{R} : y \geq 0\}$ ✓

(ii) determine the largest possible set of values $x > 0$ for which $(f \circ g)(x)$ is invertible, and give the formula for $(f \circ g)^{-1}(x)$. [3]



largest set of values : $x \geq 1$ ✓

if $y = \sqrt{|x| - 1}$

$y = \sqrt{x-1}$ (as $x \geq 0\}$ ✓

$x = \sqrt{y-1}$

$y = x^2 + 1$

$\therefore (f \circ g)^{-1}(x) = x^2 + 1$ ✓

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Question 7 (6 marks)

A particle moves with velocity $\boldsymbol{v}(t)$ at time t seconds given by

$$\boldsymbol{v}(t) = (-2\,sin\,2t)\,\boldsymbol{i} + (2\,cos\,2t)\,\boldsymbol{j}$$

and has initial position $3\boldsymbol{i} + 5\boldsymbol{j}$.

(i) Determine the object's acceleration $\boldsymbol{a}(t)$. [1]

$\underset{\sim}{a}(t) = (-4\cos 2t)\underset{\sim}{i} - (4\sin 2t)\underset{\sim}{j}$ ✓

(ii) Determine the object's position vector $\boldsymbol{r}(t)$. [3]

$\underset{\sim}{r}(t) = (\cos 2t)\underset{\sim}{i} + (\sin 2t)\underset{\sim}{j} + \underset{\sim}{c}$ ✓

$\langle 3, 5 \rangle = \langle 1, 0 \rangle + \underset{\sim}{c}$ ✓

$\underset{\sim}{c} = \langle 2, 5 \rangle$

$\therefore \quad \underset{\sim}{r} = (\cos 2t + 2)\underset{\sim}{i} + (\sin 2t + 5)\underset{\sim}{j}$ ✓

(iii) Find the Cartesian equation of the object's path. [2]

$x = \cos 2t + 2 \qquad y = \sin 2t + 5$ ✓

$(x-2)^2 = \cos^2 2t \qquad (y-5)^2 = \sin^2 2t$

$(x-2)^2 + (y-5)^2 = 1$ ✓

Question 8 (6 marks)

A line has equation $\quad r = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$

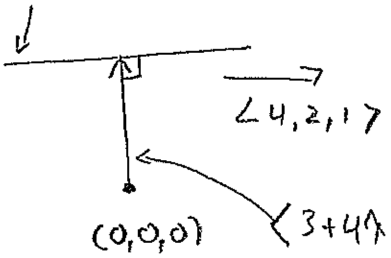
(i) Determine the Cartesian equation of the line. [3]

$x = 3 + 4\lambda \qquad y = -1 + 2\lambda \qquad z = \lambda$

$$\therefore \quad \frac{x-3}{4} = \frac{y+1}{2} = z$$

(ii) Find the point on the line closest to $(0, 0, 0)$. [3]

$L: \langle 3 + 4\lambda, -1 + 2\lambda, \lambda \rangle$



$\langle 3 + 4\lambda, -1 - 2\lambda, \lambda \rangle - \langle 0, 0, 0 \rangle$

$\therefore \quad \langle 3 + 4\lambda, -1 + 2\lambda, \lambda \rangle \cdot \langle 4, 2, 1 \rangle = 0$

$12 + 16\lambda - 2 + 4\lambda + \lambda = 0$

$21\lambda = -10$

$\lambda = -10/21$

Question 9 (7 marks)

Given the line $r = (2,-2,1) + \lambda(-1,0,3)$,

(i) find the co-ordinates of the points A and B on the line corresponding to $\lambda = -1$ and $\lambda = 1$ respectively, [1]

$\lambda = -1 \; : \; \langle 3, -2, -2 \rangle$ point A

$\lambda = 1 \; : \; \langle 1, -2, 4 \rangle$ point B.

(ii) determine the vector equation of the sphere of which AB is a diameter, [2]

$\overrightarrow{AB} = \langle -2, 0, 6 \rangle$

$\therefore$ centre is $\langle 3, -2, -2 \rangle + \frac{1}{2}\langle -2, 0, 6 \rangle$

$= \langle 2, -2, 1 \rangle$

Diameter $= |\overrightarrow{AB}|$

$= \sqrt{4+36}$

$= \sqrt{40}$

$= 2\sqrt{10}$

radius $= \sqrt{10}$

$$\left| \underset{\sim}{r} - \langle 2, -2, 1 \rangle \right| = \sqrt{10}$$

(iii) find a vector perpendicular to $(-1, 0, 3)$,
(Hint: let the vector be (a, b, c)) [2]

$$\langle a, b, c \rangle \cdot \langle -1, 0, 3 \rangle = 0$$

$$-a + 3c = 0$$

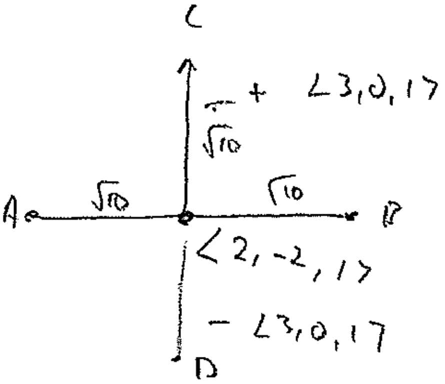
$$a = 3c$$

let $b = 0$, $a = 3$, $c = 1$

∴ a vector is $\langle 3, 0, 1 \rangle$

(iv) hence determine points C and D on the sphere in part (ii) such that CD is also a diameter and $CD \perp AB$. [2]

$$| \langle 3, 0, 1 \rangle | = \sqrt{10}$$



∴ $\langle 2, -2, 1 \rangle \pm \langle 3, 0, 1 \rangle$

∴ $\langle 5, -2, 2 \rangle$ and $\langle -1, -2, 0 \rangle$

Question 10 (5 marks)

Consider the system of equations:

$$\begin{aligned} x-2y+z &= 5 \\ ax+ay+z &= 1 \\ x+y+z &= b \end{aligned}$$

where a and b are constants.

Find all values of a and b for which the system has

(i) a unique solution, [2]

① and ③ are not parallel. Thus, if ② is not parallel to ① and ③, there will be a unique solution.

$\therefore \quad a \neq 1, \quad b \in \mathbb{R}.$

(ii) infinitely many solutions, [2]

if a is 1 and b is 1 then ② and ③ are the same plane. There will be a line of solutions.

$\therefore \quad a = 1, \quad b = 1$

(iii) no solution. [1]

if $a = 1$, $b \neq 1$. Two parallel (non equal) planes cut by a third, non-parallel plane.

Question 11 (5 marks) CA

Find a vector normal to the plane

(i) whose Cartesian equation is $2x - 3y + 5z = 11$, [1]

$\langle 2, -3, 5 \rangle$

(ii) whose vector equation is $r = \begin{pmatrix} 4 - \lambda + \mu \\ -1 + 2\lambda - \mu \\ 6 + 3\lambda + 2\mu \end{pmatrix}$ [2]

$\underset{\sim}{r} = \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\lambda + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\mu.$

$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix}$

(iii) containing the points whose position vectors are $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. [2]

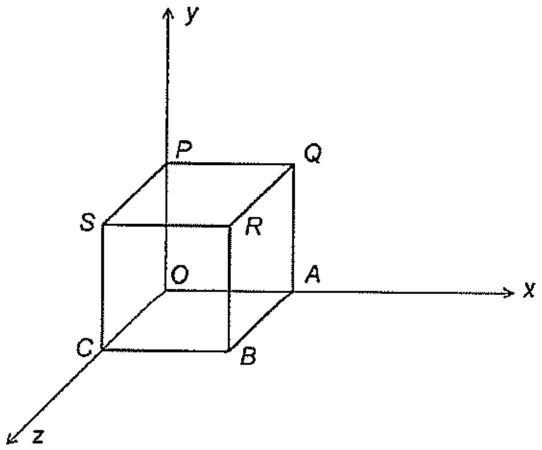
$A\ (1, 0, 0) \quad B\ (0, 1, 0) \quad C\ (0, 0, 1).$

$\overrightarrow{AB} = (-1, 1, 0) \quad \overrightarrow{AC} = (-1, 0, 1)$

$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, 1, 1 \rangle.$

Question 12 (7 marks) CA

OABCPQRS is a unit cube, as shown in the diagram:



(i) In terms of the basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, write down the position vectors of the points O, R, A and S. [2]

$O: \quad 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{0}$

$R: \quad \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$A: \quad \underset{\sim}{i}$

$S: \quad \underset{\sim}{j} + \underset{\sim}{k}$

(ii) Hence show that the diagonals OR and AS bisect each other. [3]

Midpoint of OR $= \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} - \underset{\sim}{0})$

$= \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

Midpoint of AS $= \underset{\sim}{i} + \frac{1}{2}(\underset{\sim}{j} + \underset{\sim}{k} - \underset{\sim}{i})$

$= \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

= Midpoint of QR

∴ OR and AS bisect each other.

(iii) Find the acute angle of intersection between the diagonals OR and AS, accurate to 0.01°. [2]

$\langle 1, 1, 1 \rangle, \langle -1, 1, 1 \rangle$

in CAS

Angle = 70.53°.

Question 13 [2, 3, 4 marks]

(a) Given that $p = \cos(xy)$, find an expression for $\frac{dp}{dx}$ in terms of x and y.

Hint: Let $t = xy$ and use chain rule.

$$\frac{dp}{dx} = -\sin(xy) \cdot \left(\frac{dy}{dx} x + y\right)$$

(b) Given that $y = x^{\sin y}$, find an expression for $\frac{dy}{dx}$ in terms of x and y.

$$\ln y = \sin y \ln x$$

let $y' = dy/dx$.

$$\frac{y'}{y} = (\cos y) y' \ln x + \frac{\sin y}{x}$$

$$\frac{y'}{y} - (\cos y)(\ln x) y' = \frac{\sin y}{x}$$

$$y' = \frac{\sin y}{x\left(\frac{1}{y} + \ln x \cos y\right)}.$$

(c) A curve is defined implicitly by $xy^2 = 3x^2 - 2x\sqrt{y}$. Find the gradient of the tangent to the curve where $y = 4$ and $x > 6$.

$$xy^2 = 3x^2 - 2x\sqrt{y}$$

when $y = 4$

$$x \cdot 16 = 3x^2 - 4x$$

$$0 = 3x^2 - 20x$$

$$0 = x(3x - 20)$$

$$x = \cancel{0}, \frac{20}{3} \quad (\text{as } x > 6)$$

$$y^2 + 2xy \cdot \frac{dy}{dx} = 6x - 2\sqrt{y} - \cancel{2}x \cdot \frac{1}{\cancel{2}} y^{-\frac{1}{2}} \cdot \frac{dy}{dx}$$

$$16 + 2\left(\frac{20}{3}\right)(4) \cdot y' = 40 - 4 - \frac{20}{3(2)} y'$$

$$\frac{160y'}{3} + \frac{10y'}{3} = 20$$

$$170y' = 60$$

$$y' = \frac{6}{17}$$

$$\therefore \frac{dy}{dx} = \frac{6}{17}$$

Question 14 [3, 3, 3 marks]

Find the indefinite integrals:

(a) $\int 2\sin^3 2x\,dx$

$$= \int 2\sin 2x\left(1-\cos^2 2x\right)dx$$

$$= \int 2\sin 2x - 2\sin 2x(\cos 2x)^2\,dx$$

$$= -\cos 2x - \frac{\cos^3 2x}{3} + C$$

(b) $\int \frac{2x-3}{x^2+4x-5}\,dx$

using partial fractio $\quad \frac{A}{(x+5)} + \frac{B}{(x-1)} = \frac{Ax - A + Bx + 5B}{x^2+4x-5}$

$$\therefore \begin{aligned} A + B &= 2 \\ -A + 5B &= -3 \quad + \\ 6B &= -1 \\ B &= -\frac{1}{6} \end{aligned}$$

$$\therefore A = 2 + \frac{1}{6} = \frac{13}{6}$$

$$\therefore \int \frac{13}{6(x+5)} + \frac{-1}{6(x-1)}\,dx$$

$$= \frac{13\ln|x+5|}{6} - \frac{\ln|x-1|}{6} + C.$$

(c) $\int \frac{e^{-2x} - e^{2x}}{e^{2x} + e^{-2x}} dx$

let $u = e^{2x} + e^{-2x}$

$$\frac{du}{dx} = 2e^{2x} - 2e^{-2x}$$

$$dx = \frac{du}{2e^{2x} - 2e^{-2x}}$$

$$\therefore \int \frac{e^{-2x} - e^{2x}}{u} \cdot \frac{du}{2(e^{2x} - e^{-2x})}$$

$$- \frac{1}{2} \int \frac{du}{u}$$

$$= - \frac{\ln |e^{2x} + e^{-2x}|}{2} + C$$

Question 15 [4 marks]

Evaluate $\int_1^{\frac{5}{2}} 3x\sqrt{2x-1}\,dx$

let $u = 2x - 1$ $\therefore$ $x = \frac{u+1}{2}$

$\frac{du}{dx} = 2$

$(dx = \frac{du}{2})$

$x = 1, u = 1$

$x = \frac{5}{2}, u = 4$

$$\therefore 3\int_{u=1}^{u=4} \frac{u+1}{2}\sqrt{u} \cdot \frac{du}{2}$$

$$= \frac{3}{4}\int_{u=1}^{u=4} u^{3/2} + u^{1/2}\ du$$

$$= \frac{3}{4}\left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right]_{u=1}^{u=4}$$

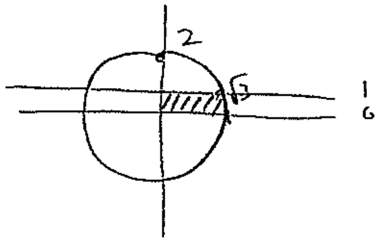
$$= \frac{3}{4}\left[\left(\frac{64}{5} + \frac{16}{3}\right) - \frac{2}{5} - \frac{2}{3}\right]$$

$$= \frac{3}{4}\left(\frac{62}{5} + \frac{14}{3}\right)$$

$$= \frac{64}{5}.$$

Question 16 [3, 4, 4 marks] CA

(a) Find the volume generated when the region enclosed by the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$ and the line $y=1$ above the x-axis is rotated 360⁰ about the y-axis.

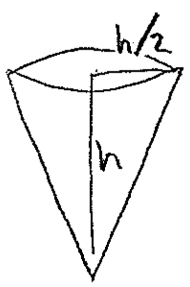


$$\boxed{x^2 = \frac{-3y^2}{4} + 3}$$

$$= \pi\int_0^1 \left(-\frac{3y^2}{4}+3\right) dy$$

$$= \frac{11\pi}{4} \quad \text{units}^3$$

(b) Water is pour into an inverted cone at a rate of $5\,cm^3s^{-1}$. If the height of the cone is twice the radius of its base, what would be the rate of increase in the depth of the water level measured from the vertex at the instant when the depth of the water is 10 cm?



$$V = \frac{\pi\left(\frac{h}{2}\right)^2 \cdot h}{3}$$

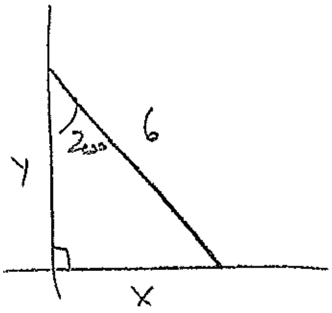
$$\left.\frac{dV}{dt}\right|_{h=10} = 25\pi$$

$$5 = 25\pi \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{5\pi} \quad cm/s$$

(c) A 6m ladder is resting against a vertical wall. If the base of the ladder is sliding outwards (away from the wall) at a constant rate of $0.02\,ms^{-1}$, what would be the rate of change of the height of the ladder at the instant when the ladder makes an angle of 20^0 with the wall?

$\frac{dx}{dt} = 0.02$ m/s.



$y = \sqrt{36 - x^2}$

$\sin 20° = \frac{x}{6}$

$x = 6 \sin 20°$

$\frac{dy}{dx}\Big|_{x = 6\sin 20°} = -0.36397$

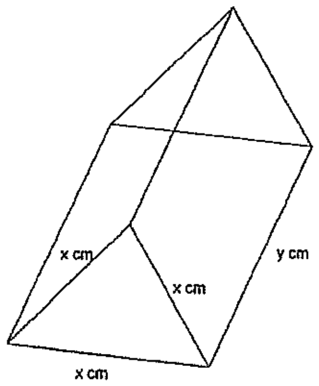
$\therefore \frac{dy}{dt} = -0.36397 \times 0.02$

$= -0.007$

$\therefore$ decreasing at 0.007 m/s

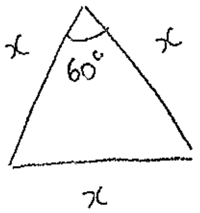
Question 17 [2, 2, 2 marks] CA

The diagram shows a triangular prism with equilateral triangles at both ends.



If the volume of the prism is 2400 cm³,

(a) establish a relationship between x and y



$$2400 = \frac{1}{2} x^2 \cdot \sin 60^\circ \, y$$

$$y = 3200\sqrt{3}\, x^{-2}$$

(b) show that the total surface area A is given by, $A = \frac{\sqrt{3}}{2}x^2 + \left(\frac{9600\sqrt{3}}{x}\right)$

$$A = \frac{x^2 \cdot \sqrt{3}}{2} + 3xy$$

$$A = \frac{\sqrt{3}x^2}{2} + \frac{9600\sqrt{3}}{x}$$

(c) use the incremental formula to find the approximate change in the total surface area if x is increased from 4 cm to 4.01 cm.

$$\left.\frac{dA}{dx}\right|_{x=4} = -596\sqrt{3}$$

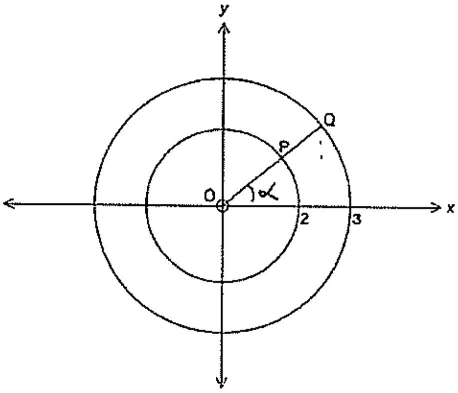
$$\delta A \sim -596\sqrt{3} \times 0.01$$

$$\sim \frac{-149\sqrt{3}}{25}$$

$$\sim -10.32 \text{ cm}^2$$

Question 18 [2, 3 marks] CA

The diagram shows a Cartesian plane with two concentric circles with radius 2 and 3 units.



P and Q are points that move on the circle with radius of 2 units and 3 units respectively. O, P and Q are collinear at any time. Point T moves inside the region between the two circles in such a way that its x-coordinate is the same as Q's and its y-coordinate is the same as P's. Assume that the line OPQ makes an angle of α radians with the positive x-axis.

(a) Find the parametric equations of the locus of T in terms of α.

$x = 3\cos\alpha$

$y = 2\sin\alpha$

(b) Find the gradient of the tangent to the locus of T when $\alpha = \frac{\pi}{3}$.

$$\frac{dx}{d\alpha} = -3\sin\alpha$$

$$\frac{dy}{d\alpha} = 2\cos\alpha$$

$$\frac{dy}{dx} = 2\cos\alpha \cdot \frac{1}{-3\sin\alpha}$$

$$= 1 \times \frac{1(2)}{-3\sqrt{3}}$$

$$= \frac{2}{-3\sqrt{3}}$$

OR $$\frac{-2\sqrt{3}}{9}$$

Question 19 (5 marks) CA

An object suspended on the end of a spring oscillates in Simple Harmonic Motion about its mean position with a frequency of 4 cycles per second and an amplitude of 5 cm.

(i) What is the exact speed of the object as it passes through its mean position? [3]

$\frac{K}{2\pi} = 4$, $K = 8\pi$, $A = 5$

$$V^2 = (8\pi)^2(5^2 - 0^2)$$

$$V = \pm 40\pi$$

speed = 40π cm/s.

(v) What is the exact distance travelled by the object in 1 second? [2]

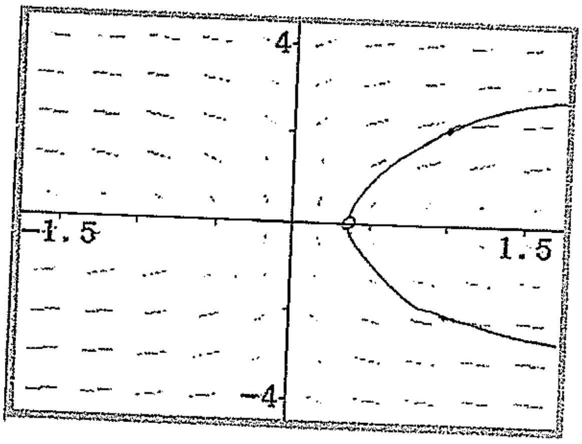
$4 \times 5 \times 4 = 80$ m

As $5 \uparrow \downarrow$

$\downarrow \uparrow \times 4$

Question 20 (6 marks)

The first-order differential equation, $\frac{dy}{dx} = \frac{1}{xy}$, has a slope field shown in the diagram below.



(a) Sketch the particular solution which passes through the points (1, 2) and (1, -2). [2]

(b) Determine the equation of the curve sketched in part (b). [4]

$$\frac{dy}{dx} = \frac{1}{xy}$$

$$\int y\,dy = \int \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \ln|x| + c$$

$x = 1, \; y = 2$

$$2 = \ln|1| + c$$

$$c = 2$$

$$\frac{y^2}{2} = \ln|x| + 2$$

$$y^2 = 2\ln|x| + 4$$

$$y^2 = 2\ln(x) + 4$$

Question 21 (6 marks) CA

A biologist applies the logistic model to the growth of bacteria in an experiment. She models the population $P(t)$ after t minutes according to the differential equation

$$\frac{dP}{dt} = \frac{P}{1000}\left(3 - \frac{P}{10}\right)$$

P_0.

where P is in millions. The initial population is $\underline{100\,000}$.

(a) What is the growth rate of the bacteria when the population reaches 1 million? [2]

$$\frac{dP}{dt} = \frac{1}{1000}\left(3 - \frac{1}{10}\right)$$

$= 0.0029$ million ppl/year.

(b) What is the population after one hour? [3]

$$\frac{dP}{dt} = 0.0001\,P(30 - P)$$

$$P = \frac{30 \times 0.1}{0.1 + (29.9)e^{-0.003t}}$$

$$P = \frac{30}{1 + 299e^{-0.003t}}$$

∴ when $t = 60$

$P = 0.1196$

∴ $\sim 120\,000$

(c) What is the limiting population? [1]

30.

Tip: Use Formula sheet.

Question 22 (7 marks) CA

A weight on the end of a spring is oscillating. It's displacement, x metres, from the mean position at time t seconds is given by $x = 3\sin(kt + \frac{\pi}{3})$, where $k > 0$.

(a) Show that the weight is moving with simple harmonic motion. [2]

$$\frac{dx}{dt} = 3K\cos\left(Kt + \frac{\pi}{3}\right)$$

$$\frac{d^2x}{dt^2} = -3K^2\sin\left(Kt + \frac{\pi}{3}\right)$$

$$= -K^2 x$$

$\therefore$ The body's motion is simple harmonic

(b) Determine the value of k given that the weight has an acceleration of $a = (-15x)\ m/s^2$. [2]

$$\therefore 15 = K^2$$

$$K = -\cancel{\sqrt{15}}, \sqrt{15}$$

$$K = \sqrt{15} \quad \text{as} \quad K > 0$$

(c) Determine the distance travelled by the weight during the third second. [3]

$$\int_2^3 |v(t)|\,dt = \int_2^3 \left|3\sqrt{15}\cos\left(\sqrt{15}t + \frac{\pi}{3}\right)\right| dt$$

$$= 8.07\text{m}.$$

Question 23 (4 marks) CA

If X is a Binomial random variable with probability of success p in each of n trials, then:

$$\bar{X} = np$$

and $$s = \sqrt{np(1-p)}\,.$$

A fair die is rolled 12 times. Let X = the number of sixes rolled.

(a) Determine $\bar{X}$ and s exactly. [2]

$$\bar{X} = 12 \times \frac{1}{6}$$

$$= 2$$

$$s = \sqrt{12\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)}$$

$$= \frac{\sqrt{15}}{3}$$

Suppose that the above experiment is carried out 40 times (each time with 12 rolls of the die).

Let Y = the average number of sixes rolled per trial over the 40 trials.

(b) Use an appropriate normal distribution to find the probability (accurate to 4 decimal places) that Y is less than 1.8. [2]

$$Y \sim N\left[2\ ,\ \left(\frac{\sqrt{15}}{3\sqrt{(40)}}\right)^2\right]$$

$$P(Y < 1.8) \sim 0.1636$$

Question 24 (4 marks) CA

In a dairy, a machine produces blocks of butter purported to weigh 500 g.

The owners suspect that the mean weight of the blocks is not actually 500 g and measure a random sample of 300 blocks, yielding a mean of 498 g and a standard deviation of 2.4 g.

$\bar{x}$ s

(a) Determine a 95% confidence interval for the true mean weight (correct to 2 decimal places). [2]

$$498 \pm 1.960 \times \frac{2.4}{\sqrt{300}}$$

$$(497.73 < \mu < 498.27)$$

* Can be done in CAS

- One Sample Z int.

(b) How many blocks should be in a sample in order to be 95% confident that the true mean is within 0.100 g of the sample mean? [2]

$$1.960 \times \frac{2.4}{\sqrt{n}} \leq 0.1$$

$$n \geq 2212.8$$

$$\therefore n = 2213$$

Question 25 (5 marks) CA (0, 0)

Consider the curve that passes through the origin and satisfies

$$\frac{dy}{dx} = e^{x+y} \quad .$$

(a) Prove that $e^x - 2 = -e^{-y}$. [3]

$$\frac{dy}{dx} = e^x e^y$$

$$\int \frac{dy}{e^y} = \int e^x dx$$

$$-e^{-y} = e^x + c$$

use (0, 0)

$$-1 = 1 + c$$

$$c = -2$$

$$\therefore -e^{-y} = e^x - 2$$

$$\therefore e^x - 2 = -e^{-y}$$

(b) Hence show that $x < ln2$ for all points on the curve. [2]

$-e^{-y} < 0$ for all values of y

$\therefore e^x - 2 < 0$ for all values of x

$$e^x < 2$$

$$x < \ln 2$$

Question 26 (7 marks) CA

A particle moves with rectilinear motion such that $\frac{d^2x}{dt^2} = 3x$. Initially the particle has a velocity of 6 m/s and a displacement of 3 m. The velocity and displacement of the particle are always greater than 0.

(a) Determine the velocity of the particle, v, in terms of x. [3]

$$a = 3x$$

$$\therefore \quad v \cdot \frac{dv}{dx} = 3x$$

$$v = \sqrt{3(x^2+3)}$$

[using dSolve in cas]. $x=3 \quad v=6$

(b) Determine x when $t = 4$, correct to the nearest metre. [4]

$$\frac{dx}{dt} = \sqrt{3(x^2+3)}$$

[in cas 'dSolve'] using $t=0$, $x=3$

$$x = \frac{\left((2\sqrt{3}+3)^{\frac{\sqrt{3}}{3}} e^{t}\right)^{2\sqrt{3}} - 3}{2\left((2\sqrt{3}+3)^{\frac{\sqrt{3}}{3}} e^{t}\right)^{\sqrt{3}}}$$

$$x|_{t=4} = 3299$$

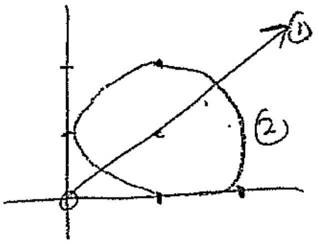
$\therefore$ 3299m

Question 27 (10 marks)

(a) By sketching appropriate diagrams, show that there are exactly two solutions to the simultaneous complex equations: $|z-1-i|=1$ and $arg(z)=\frac{\pi}{4}$.

State these solutions in exact Cartesian form. [5]

$$|z-(1+i)|=1$$



①: $y=x$

② $(x-1)^2+(y-1)^2=1$

$$(x-1)^2+(x-1)^2=1$$

$$2(x-1)^2=1$$

$$(x-1)^2=\frac{1}{2}$$

$$(x-1)=\pm\frac{1}{\sqrt{2}}$$

$$x=1+\frac{\sqrt{2}}{2},\ 1-\frac{\sqrt{2}}{2}$$

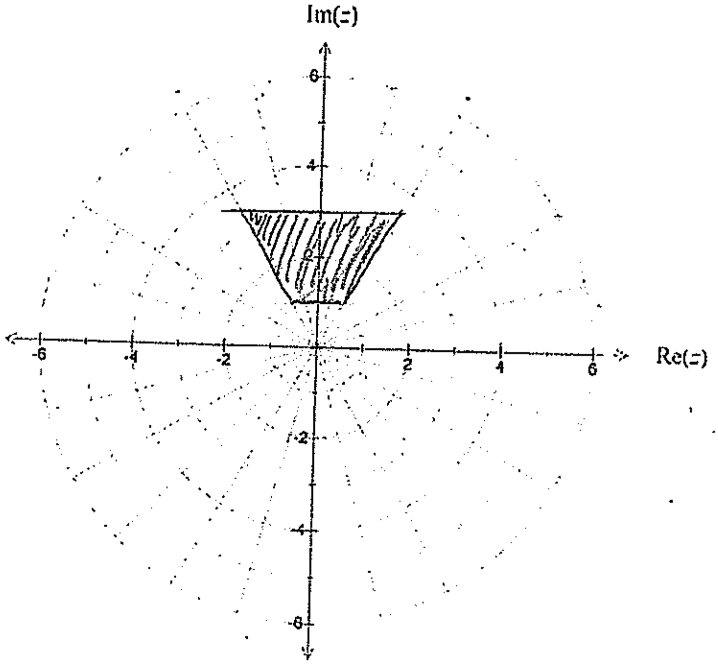
$$x=1+\frac{\sqrt{2}}{2},\ y=1+\frac{\sqrt{2}}{2}$$

and

$$x=1-\frac{\sqrt{2}}{2},\ y\ 1-\frac{\sqrt{2}}{2}$$

(b) Sketch the following set of points on the argand plane provided: [3]

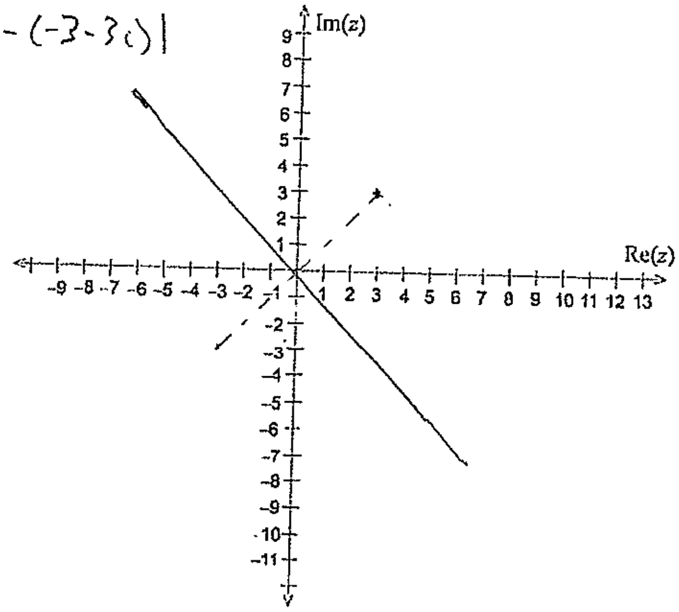
$\{z: 1 \leq Im\, z \leq 3, \frac{\pi}{3} \leq arg\, z \leq \frac{2\pi}{3}\}$



(c) Sketch the following set of points on the argand plane: [2]

$\{z: |z - 3 - 3i| = |z + 3 + 3i|\}$

$|z - (3+3i)| = |z - (-3-3i)|$



Question 28 (8 marks) CA

Bob takes a random sample of 180 'Best Brand' tyres and calculates their circumferences. The sample mean is 150 cm and the sample standard deviation is 2.5 cm.

(a) Using Bob's sample, obtain a 90% confidence interval for the population mean of the circumference of 'Best Brand tyres. [4]

$\mu = 150 \pm 1{\cdot}645 \times \frac{2{\cdot}5}{\sqrt{180}}$

$= 150 \pm 0{\cdot}3065$

$149.69 \leq \mu \leq 150.31$

(b) Greg takes a sample of 75 'Best Brand tyres and calculates their circumferences. The sample standard deviation is 3.0 cm and a confidence interval for the population mean of the circumference of 'Best Brand' tyres is found to be $149.5\ cm \leq \mu \leq 151\ cm$. Determine the confidence level of the interval correct to 3 significant figures. [4]

width $= 151 - 149.5$

$= 1{\cdot}5$

STD Error = $0{\cdot}75$

$0{\cdot}75 = z \times \frac{3{\cdot}0}{\sqrt{75}}$

$z = 2.165$

97.0% confident