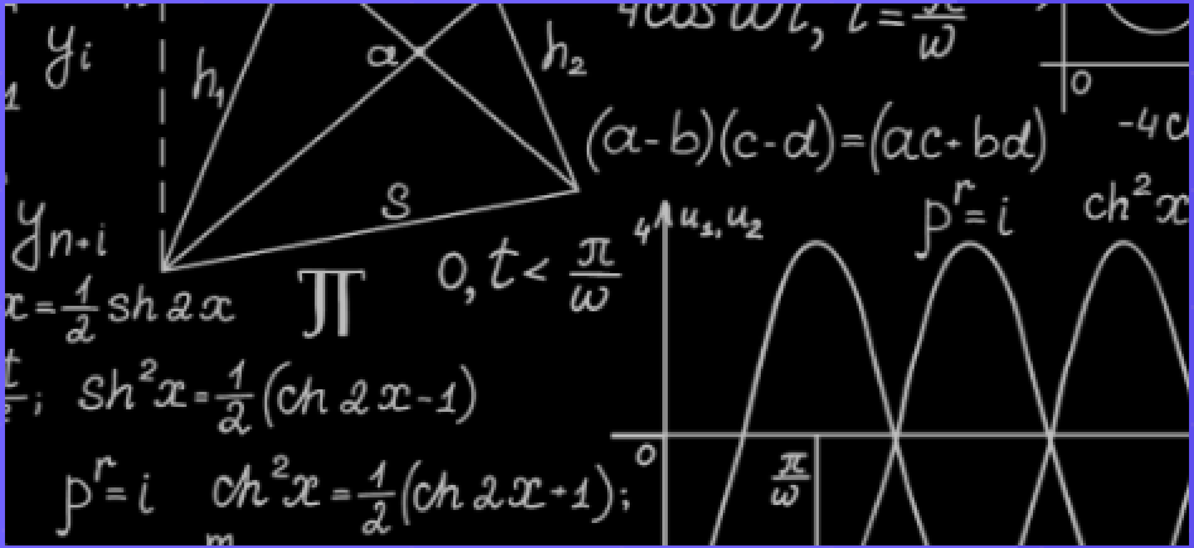
Maths Specialist 2026

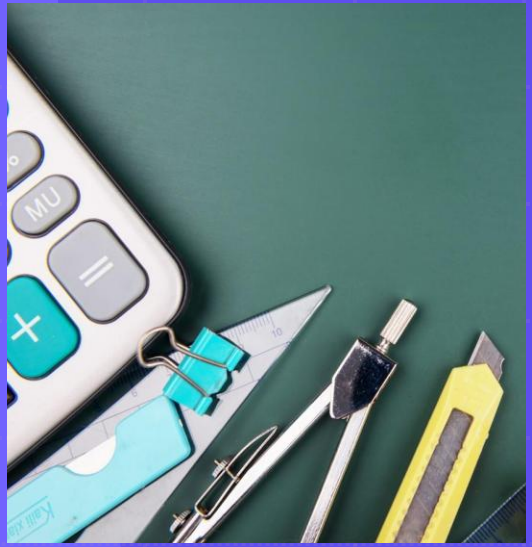
Chris Simpson

Welcome



Who am I?

- I have been teaching since 2001…. Before you were born (Wow now I feel old!)
- I have taught at a range of schools – Bullsbrook SHS, Sacred Heart College, Nagle Catholic College, Servite College, Aranmore Catholic College, Laverton DHS, Prendiville Catholic College.
- I have been Head of Maths in three schools (Nagle, Servite and Aranmore)
- I have since those days been in leadership roles in schools (Director of College Operations, Deputy Principal)
- For the last two years I have been teaching from home and working with Catholic Education in the ViSN program teaching to students in schools where a Specialist class can't be run.
- I will continue to teach with ViSN next year as well.



Agenda

	Syllabus Points	Time(min)
Introductions and welcome		15
Unit 3 - Topic 1 – Complex numbers	3.1.1 – 3.1.15	20
Unit 3 - Topic 2 – Functions and graph sketching	3.2.1 - 3.2.8	20
Break time		10
Unit 3 - Topic 3 – Vectors in three dimensions	3.3.1 – 3.3.15	20
Unit 4 - Topic 1 – Int and applications of integration	4.3.1 – 4.3.7	20
Break time		10

Agenda

	Syllabus Points	Time(min)
Unit 4 - Topic 2 – Rates of change and diff eq	4.2.1 – 4.2.7	20
Unit 4 - Topic 3 – Statistical inference	4.3.1 - 4.3.7	20
Break time		10
WACE Exam - the details/tips		10
WACE questions		40
What's next/finishing up		15

What is the course?

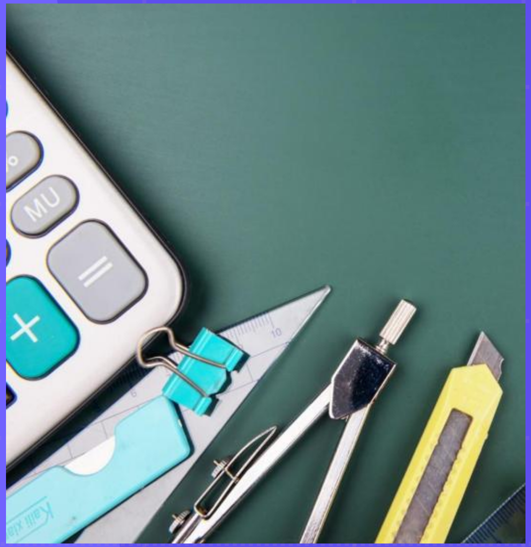
The only document you really need now, is the syllabus.

Not the textbook, not the teacher's notes, not the videos online, not WACE Vault!

The examiners were given the syllabus, maybe some suggestions/areas to emphasise, the previous year's exam reports and told "go write an exam for the course".

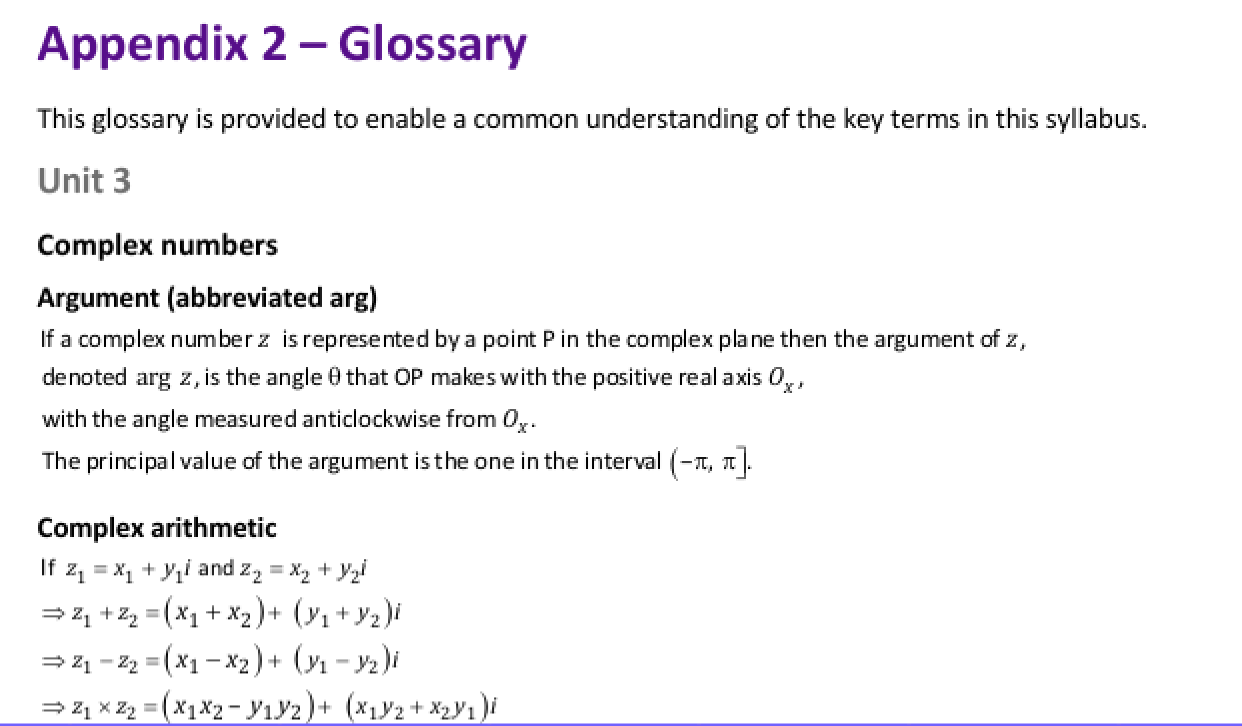
They didn't consult your teacher, textbooks, WACE Vault, Snapchat, etc

The course is the syllabus



Glossary? This isn't English!

But we (well you….) and the examiners need to be talking the same language. To get around differing interpretation of words, there is a glossary with each subject's syllabus. It is at the back of the syllabus (in a dark place you might never have seen!)



Appendix 2 – Glossary

This glossary is provided to enable a common understanding of the key terms in this syllabus.

Unit 3

Complex numbers

Argument (abbreviated arg)

If a complex number z is represented by a point P in the complex plane then the argument of z, denoted $\arg z$, is the angle θ that OP makes with the positive real axis O_x, with the angle measured anticlockwise from O_x.

The principal value of the argument is the one in the interval $(-\pi, \pi]$.

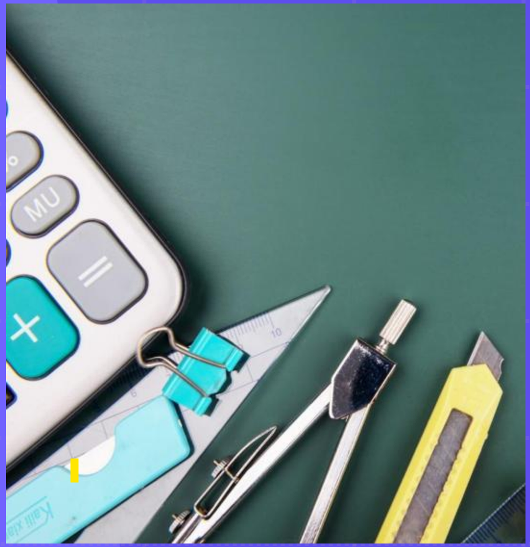
Complex arithmetic

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$\Rightarrow z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$

$\Rightarrow z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$

$\Rightarrow z_1 \times z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$

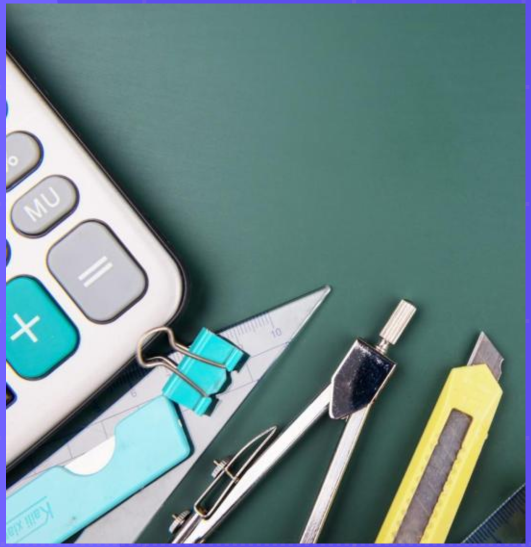


What's ahead?

We will cover each topic in the following manner

- Starting with a syllabus review
- Some teaching staying true to the syllabus points
- An exam question or two for you to do
- Questions from you?

Ready to go?



Unit 3 Topic 1

Complex numbers

3.1.1 – 3.1.15

Syllabus points

Cartesian forms

3.1.1 review real and imaginary parts Re(z) and Im(z) of a complex number z

3.1.2 review Cartesian form

3.1.3 review complex arithmetic using Cartesian forms

Complex arithmetic using polar form

3.1.4 use the modulus |z| of a complex number z and the argument Arg (z) of a non-zero complex number z and prove basic identities involving modulus and argument

3.1.5 convert between Cartesian and polar form

3.1.6 define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these

3.1.7 prove and use de Moivre's theorem

Syllabus points

The complex plane (The Argand plane)

3.1.8 examine and use addition of complex numbers as vector addition in the complex plane

3.1.9 examine and use multiplication as a linear transformation in the complex plane

3.1.10 identify subsets of the complex plane determined by relations such as

$$|z-3i| \leq 4, \ \frac{\pi}{4} \leq Arg(z) \leq \frac{3\pi}{4} \text{ and } |z-1| = 2|z-i|$$

Roots of complex numbers

3.1.11 determine and examine the nth roots of unity and their location on the unit circle

3.1.12 determine and examine the nth roots of complex numbers and their location in the complex

Syllabus points

Factorisation of polynomials

3.1.13 prove and apply the factor theorem and the remainder theorem for polynomials

3.1.14 determine and use conjugate roots for polynomials with real coefficients

3.1.15 solve simple polynomial equations

Complex numbers revision

Complex number is a number that consists of a real part and imaginary part and when written in Cartesian form

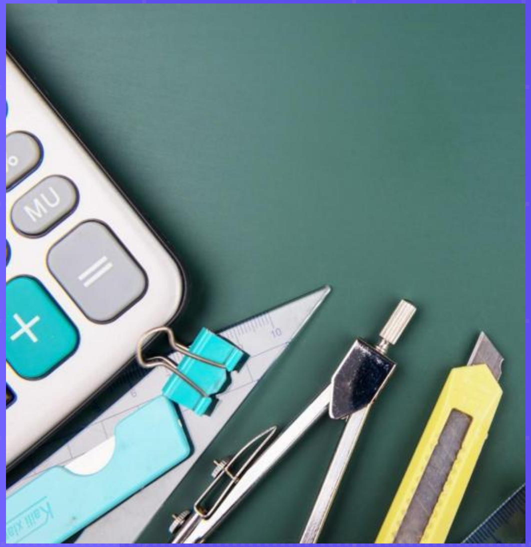
$$z = a + bi$$

a and b are real numbers and $i = \sqrt{-1}$, making the second term the imaginary part

For example, the complex number

$$z = 3 - 4i$$

has a real part, which is written as Re(z) = 3, and an imaginary part, which is written as Im(z) = -4.

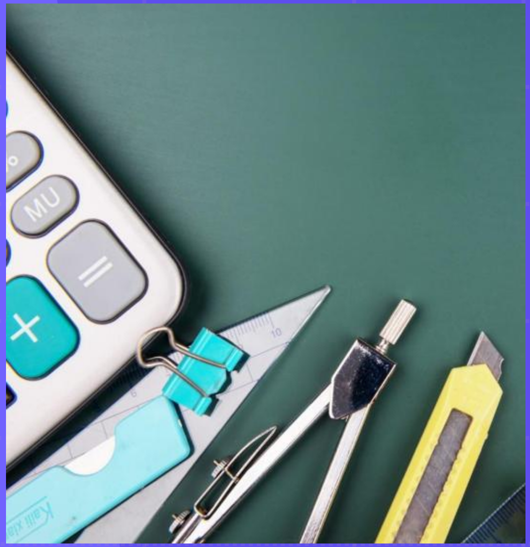


Complex numbers revision

The conjugate of a complex number is where we keep the real component the same, but change the sign of the complex component.

If $z = a + bi$, then $\bar{z} = a - bi$

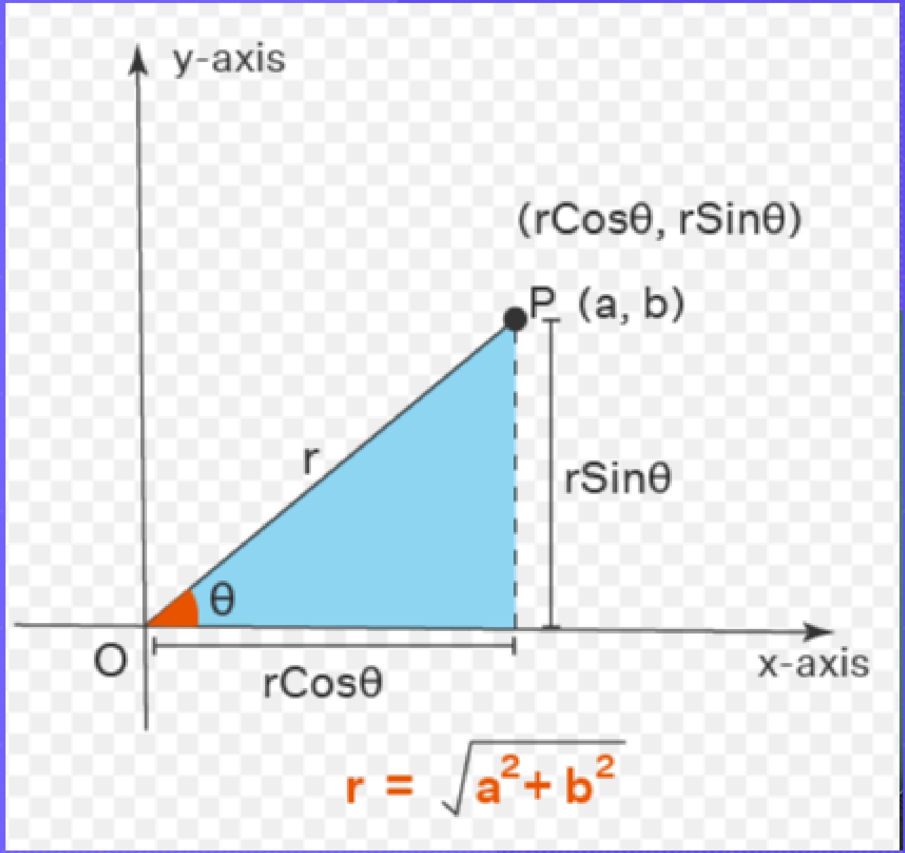
$z = a + bi$	$\bar{z} = a - bi$



Complex numbers revision

The magnitude of a complex number or modulus is the scalar value or the absolute value of its distance from the origin.

$$\text{Mod}\,(z) = |z| = \sqrt{a^2 + b^2} = r$$



The argument of a complex number is the angle it makes with the positive x-axis.

$$\text{Arg}(z) = \theta, \quad \tan\theta = \frac{b}{a}, \quad -\pi < \theta \le \pi$$

Complex numbers revision

A variety of 'identities' result from our knowledge of complex numbers that may be useful in questions.

$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$	$\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z \rvert^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\lvert z \rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

From Year 11 Formula Sheet

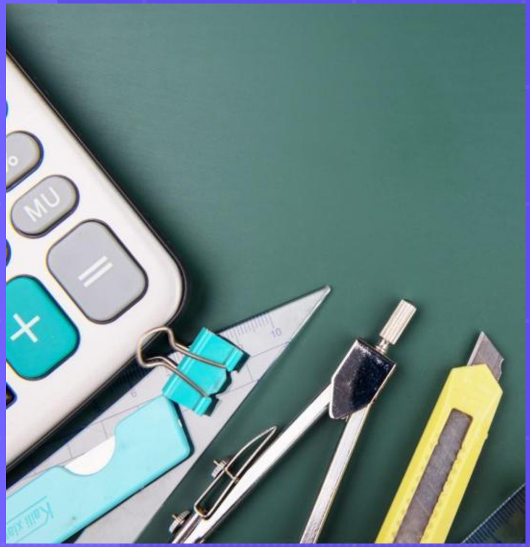
Complex Numbers

For $z = a + ib$, where $a, b \in \mathbb{R}$, $i^2 = -1$

Modulus: $\operatorname{mod} z = \lvert z \rvert = \lvert a + ib \rvert = \sqrt{a^2 + b^2}$

Product: $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$

Conjugate: $\bar{z} = a - ib$, $z\bar{z} = \lvert z \rvert^2$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$



Operations on complex numbers

Adding, subtracting and mulitplying complex numbers

You would have learn that when adding and subtracting complex numbers we use a similiar method to vector addition and subtraction in that we add/subtract the real parts and <u>separately</u> add/subtract the imaginary parts:

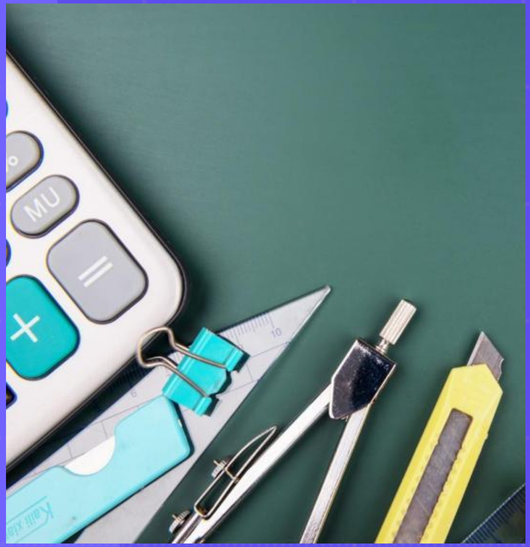
For $z = 3 + 4i, \; w = -2 - i$

Example 1

$$z + w$$
$$= (3 - 2) + (4 - i)i$$
$$= 1 + 3i$$

Example 2

$$2z - 3w$$
$$= \; 2(3 + 4i) - 3(-2 - i)$$
$$= 6 + 8i + 6 + i$$
$$= 12 + 9i$$



Operations on complex numbers

Dividing complex numbers

To divide two complex numbers we must 'realise' the denominator. That is, make the denominator only have real parts. This is done by multiplying the resulting fraction by the conjugate of the denominator (like rationalising denominators).

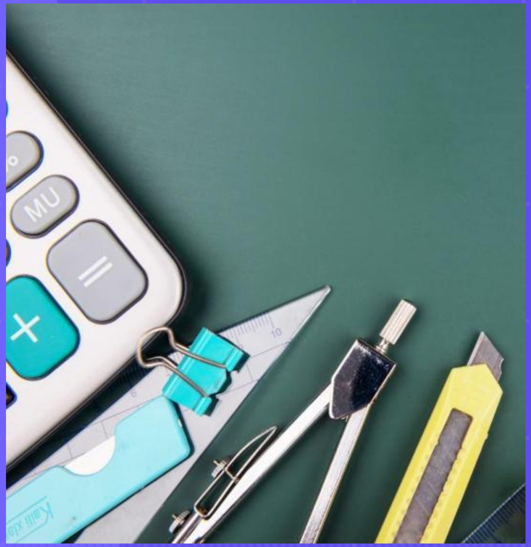
$$\frac{z}{w} = \frac{3+4i}{-2-i}$$

$$= \frac{3+4i}{-2-i} \times \frac{-2+i}{-2+i}$$

$$= \frac{-6+3i-8i-4}{4-2i+2i+1}$$

$$= \frac{-10-5i}{5}$$

$$= -2-i$$



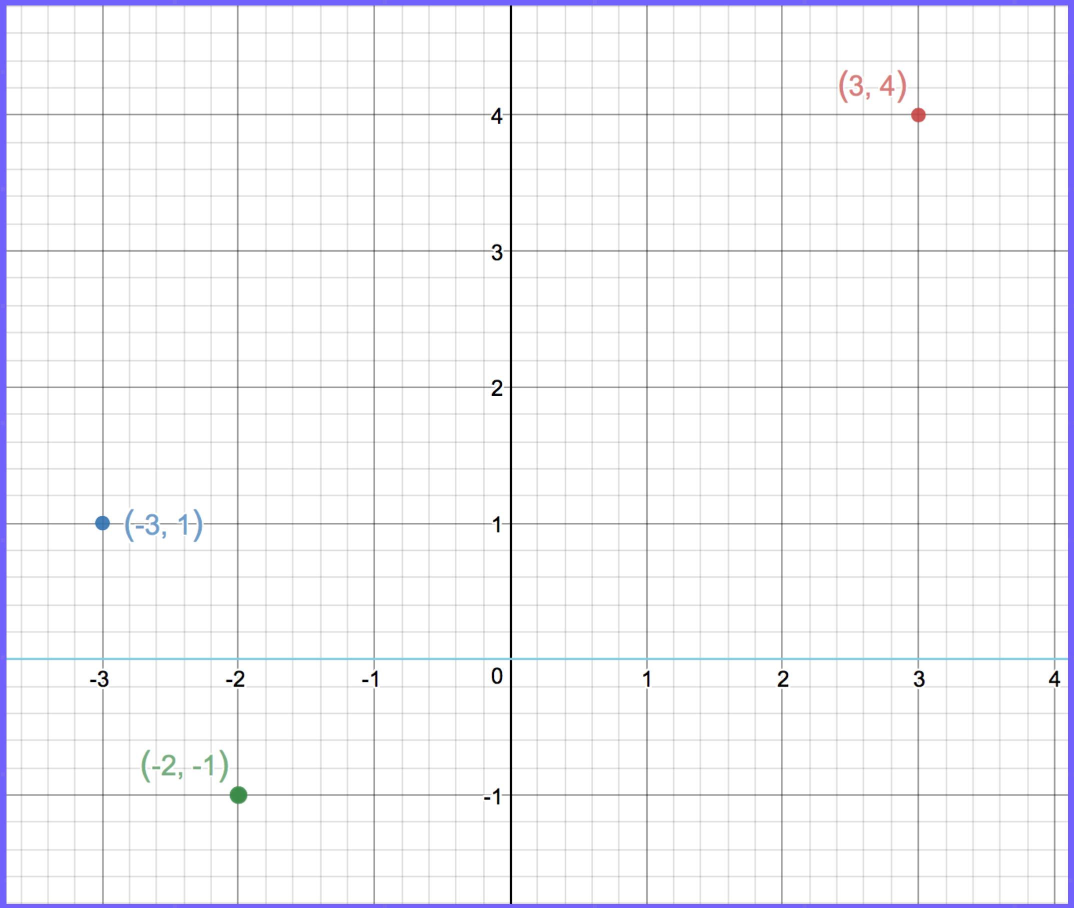
Plotting complex numbers

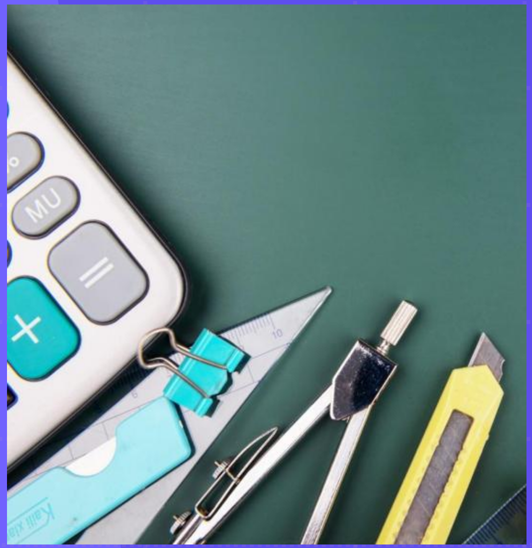
Let's take the complex number $z = a + bi$ and write it as an orded pair (as coordinates) as (a, b). From here we can plot it on the complex plane.

$z_1 = 3 + 4i$

$z_2 = -3 + i$

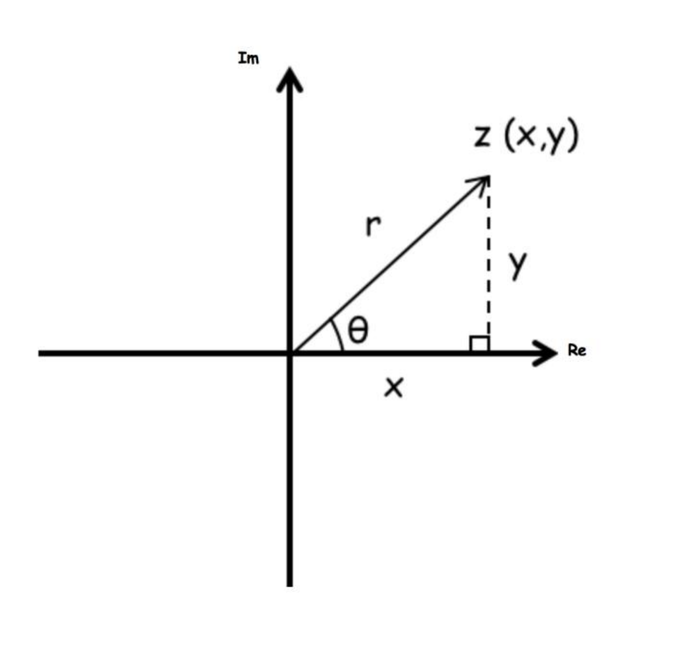
$z_3 = -2 - i$





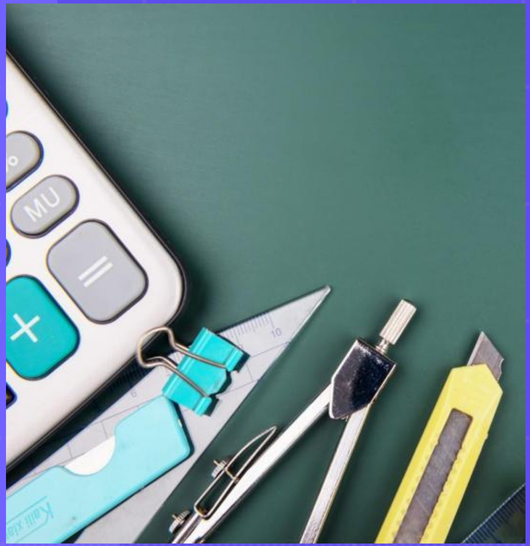
Converting forms of a complex number

Let us consider the following complex number $z = x + iy$ which is plotted on the complex plane:



The polar from of z is determining by:

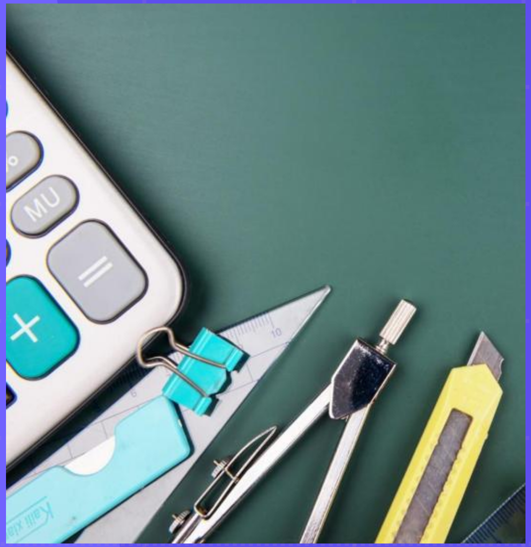
- modulus 'r' (magnitude) where $r = \sqrt{x^2 + y^2}$
- argument 'θ' (which is the direction measured from the positive Real axis). This is always measured in the domain $-\pi < \theta \leq \pi$



Converting forms of a complex number

We can also see this by letting

$$z = x + iy$$
$$z = rcos\,\theta + i\,r\sin\theta$$
$$z = r(\cos\theta + i\,sin\,\theta)$$
$$z = r\,\mathrm{Cis}\,(\theta)$$



Hence, $z = r\,\mathrm{Cis}\,(\theta)$ is the polar form of a complex number where:

- r is the modulus (same as magnitude)
- θ the argument (same as the angle the complex number is inclined from the positive real axis).

Converting forms of a complex number

We can also see this by letting

$$z = x + iy$$
$$z = r\cos\theta + i\,r\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z = r\,\mathrm{Cis}\,(\theta)$$

Hence, $z = r\,\mathrm{Cis}\,(\theta)$ is the polar form of a complex number where:

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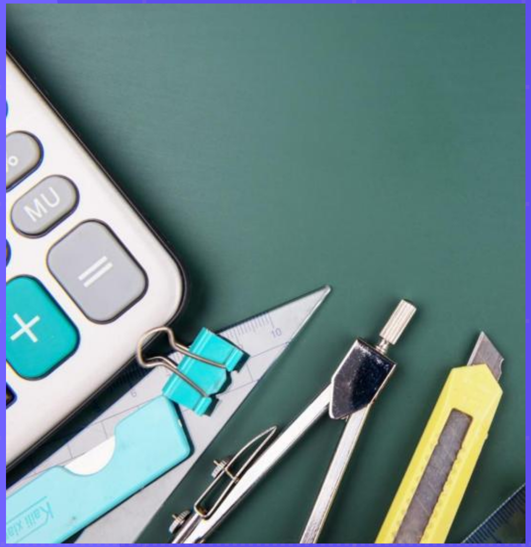


For example, $5\cos\frac{\pi}{3} + i\,5\sin\frac{\pi}{3} = 5\,cis\,\frac{\pi}{3}$

Multiplying in complex

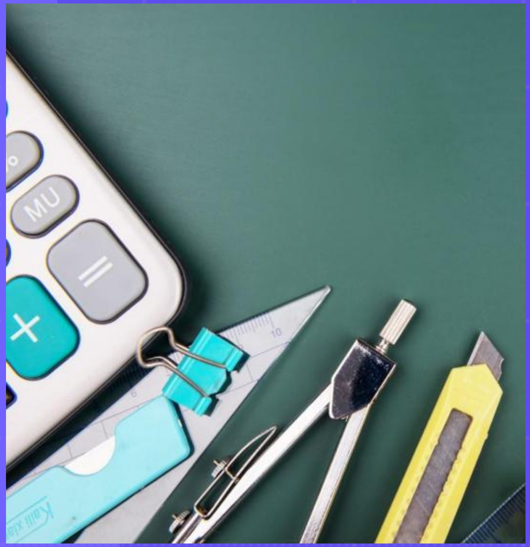
We don't want to convert back to cartesian form to perform operations, far too tedious. We can work in complex forms completely.

Firstly we recall

$$\boxed{\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2}$$

$$\boxed{\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2}$$

$$\cos^2\theta + \sin^2\theta = 1$$



Multiplying in complex

Multiplying a complex number z_1 by another complex number z_2, both in the <u>modulus-argument</u> form

$z_1 = r_1(cos\theta_1 + isin\theta_1)$ $\qquad$ $z_2 = r_2(cos\theta_2 + isin\theta_2)$

$$z_1z_2 = r_1(cos\theta_1 + isin\theta_1)\times r_2(cos\theta_2 + isin\theta_2)$$

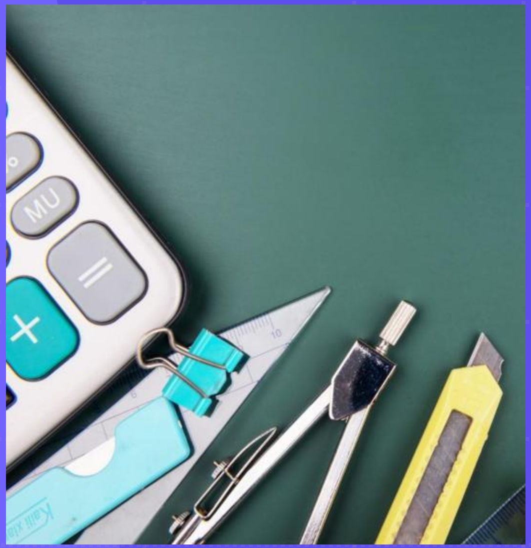
Rewrite

$$z_1z_2 = r_1r_2\boxed{(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)}$$

Now you can expand the double bracket as you would with a quadratic

$$z_1z_2 = r_1r_2(cos\theta_1cos\theta_2 + icos\theta_1sin\theta_2 + isin\theta_1cos\theta_2 + i^2sin\theta_1sin\theta_2)$$

$$z_1z_2 = r_1r_2(\boxed{cos\theta_1cos\theta_2} + \boxed{icos\theta_1sin\theta_2 + isin\theta_1cos\theta_2}\boxed{- sin\theta_1sin\theta_2})$$



Multiplying in complex

Then using the trig identities we just revised earlier

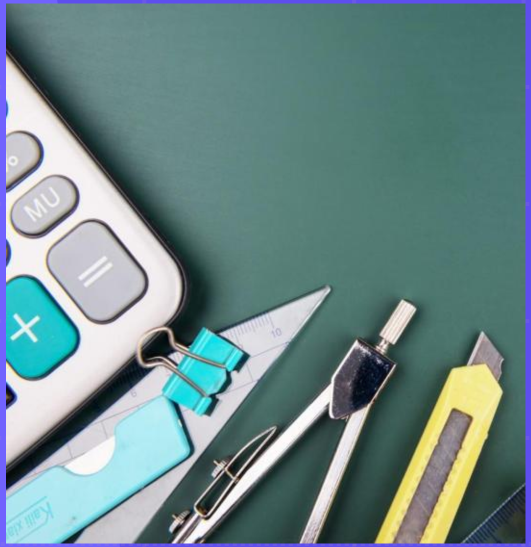
$$z_1z_2 = r_1r_2\,(\boxed{cos\theta_1cos\theta_2} + \boxed{icos\theta_1sin\theta_2 + isin\theta_1cos\theta_2}\,\boxed{- sin\theta_1sin\theta_2)}$$

$$\boxed{z_1z_2 = r_1r_2\,(\,cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))}$$

And hence we can say

$$z_1z_2 = r_1r_2\; cis\;(\theta_1 + \theta_2)$$

When we multiply two complex numbers, we multiply the moduli (plural of modulus) and add the arguments (argumenti anyone??)

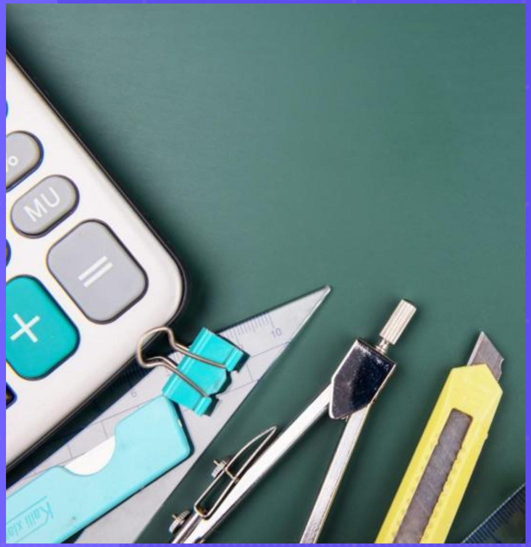


Dividing in complex

In a similar way to multiplying, but with negated versions of the trig. identities we used, we can divide complex numbers by :

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, cis\,(\theta_1 - \theta_2)$$

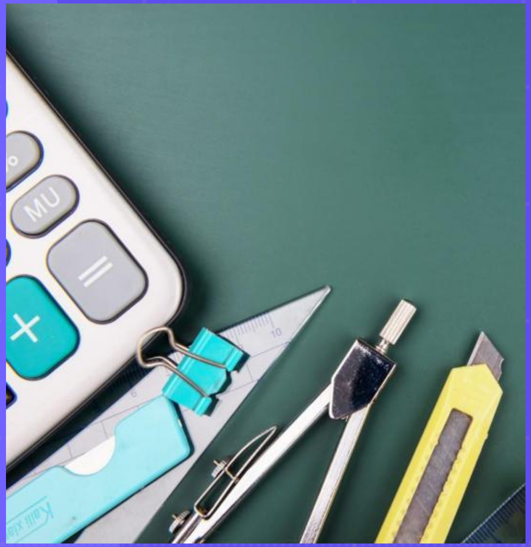
To divide two complex numbers we divide the moduli and subtract the arguments



Operations in complex

Some useful 'identities' in complex form are

Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis\,(-\theta)$
$z_1 z_2 = r_1 r_2 cis\,(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} cis\,(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \dfrac{1}{cis\,\theta}$



de Moivre's Theorem

Consider the complex number $z = r \text{ cis } \theta$. If we raise z to a positive power:

$$z^n = (r\ cis\ \theta)^n$$

$$= (r\ cis\ \theta)(r\ cis\ \theta)(r\ cis\ \theta) \ldots \ldots (r\ cis\ \theta)(r\ cis\ \theta) \text{ multiplied n times.}$$

Therefore using our knowledge of multiplication:

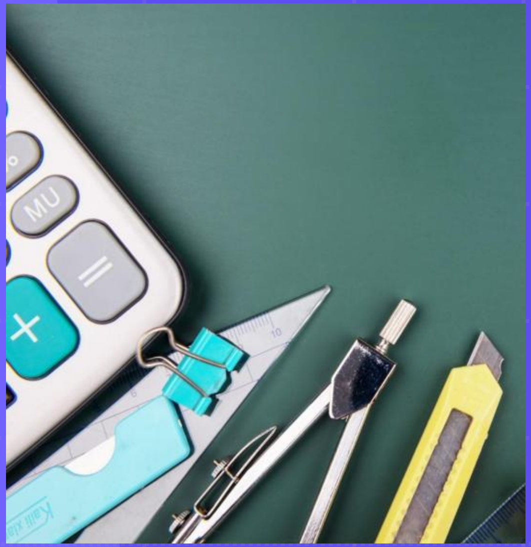
$$= r^n cis\ (n\theta)$$

Hence for all rational n,

$$(r\ cis\ \theta)^n = r^n cis\ (\mathrm{n}\theta)$$

If n is negative,

$$(r\ cis\ \theta)^{-n} = \frac{1}{r^n} cis(-n\theta)$$



Proof of de Moivre's Theorem

This is a proof by induction, where we prove for a value, then assume n = k, then prove for n = k + 1. It is a ladder style proof.

Proving that: $[r(cos\theta + isin\theta)]^n = r^n(cosn\theta + isinn\theta)$

is true for all positive integers

INITIAL STEP

→ Show that the statement is true for n = 1

$$[r(cos\theta + isin\theta)]^n = r^n(cosn\theta + isinn\theta)$$

Sub in n = 1

$$[r(cos\theta + isin\theta)]^1 = r^1(cos1\theta + isin1\theta)$$

Simplify each side

$$r(cos\theta + isin\theta) = r(cos\theta + isin\theta)$$

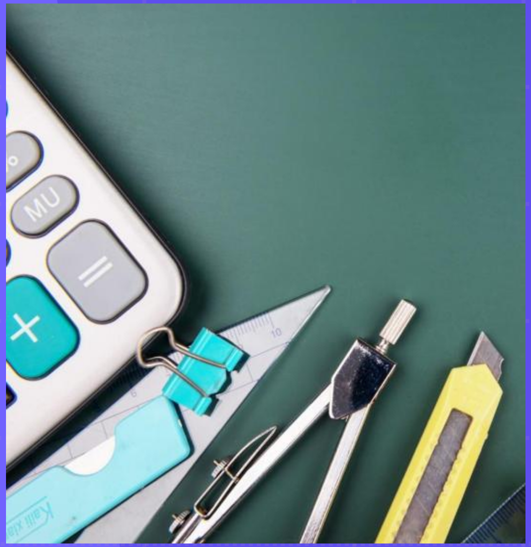
BASE STEP

→ Assume that the statement is true for n = k

$$[r(cos\theta + isin\theta)]^n = r^n(cosn\theta + isinn\theta)$$

Replace n with k

$$[r(cos\theta + isin\theta)]^k = r^k(cosk\theta + isink\theta)$$



Proof of de Moivre's Theorem

INDUCTIVE STEP

→ Show that if true for n = k, the statement is also true for n = k + 1

$$[r(cos\theta + isin\theta)]^n$$

$$[r(cos\theta + isin\theta)]^{k+1}$$

Sub in n = k + 1

You can write this as two separate parts as the powers are added together

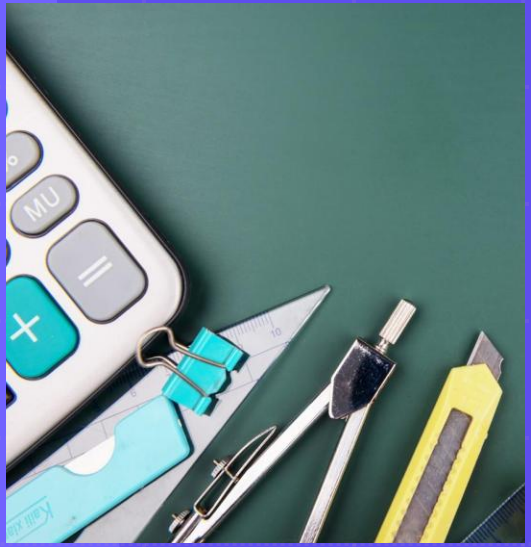
$$[r(cos\theta + isin\theta)]^k \times [r(cos\theta + isin\theta)]^1$$

We can rewrite the first part based on the assumption step, and the second based on the initial step

$$= r^k(cosk\theta + isink\theta) \times r(cos\theta + isin\theta)$$

Using the multiplication rules from earlier
→ Multiply the moduli, add the arguments

$$= r^{k+1}(\cos(k+1)\theta + isin(k+1)\theta)$$



Proof of de Moivre's Theorem

CONCLUSION

→ Explain why this shows it is true…
→ We showed the statement is true for n = 1

We then assumed the following:

$$[r(cos\theta + isin\theta)]^{\boxed{k}} = r^{\boxed{k}}(cos\boxed{k}\theta + isin\boxed{k}\theta)$$

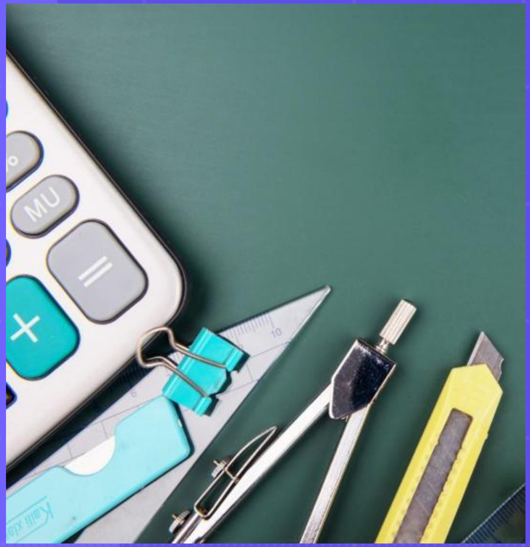
Using the assumption, we showed that:

$$[r(cos\theta + isin\theta)]^{\boxed{k+1}} = r^{\boxed{k+1}}(\cos\boxed{(k+1)}\theta + isin\boxed{(k+1)}\theta)$$

As all the 'k' terms have become 'k + 1' terms, if the statement is true for one term, it must be true for the next, and so on…

→ The statement was true for 1, so must be true for 2, and therefore 3, and so on…

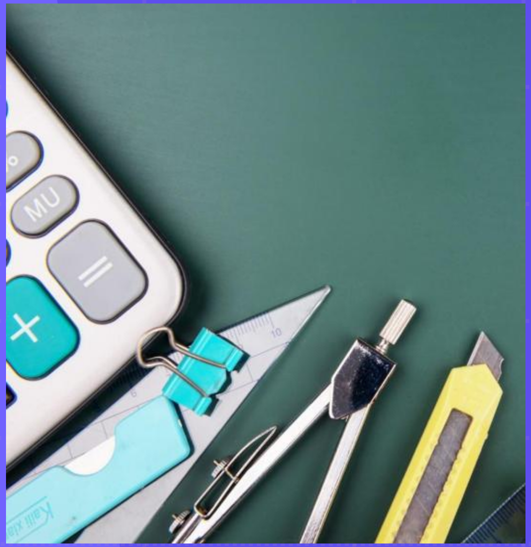
→ We have therefore proven the statement for all positive integers!



de Moivre’s Theorem

Some useful ‘identities’ for complex numbers and de Moivre's theorem are

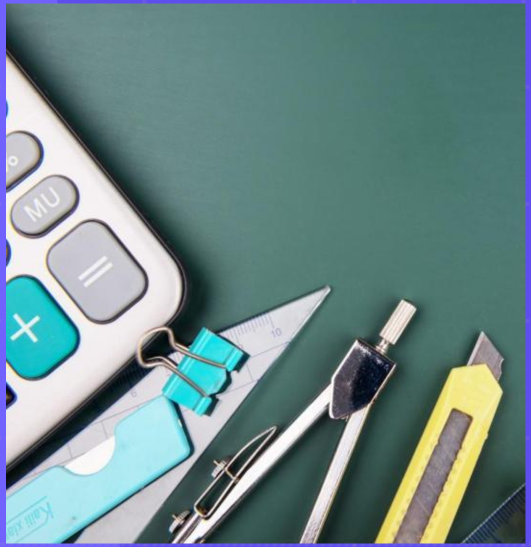
De Moivre’s theorem	
$z^n = \|z\|^n \, cis\,(n\theta)$	$(cis\ \theta)^n = \cos n\theta + i \sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\dfrac{\theta+2\pi k}{q} + i\sin\dfrac{\theta+2\pi k}{q}\right)$,	for k an integer



Complex plane

In complex numbers, the locus is a set of points (geometrically, a region, plural loci) on the complex plane that satisfies an equation or inequality.

Much like graphing linear inequalities on the Cartesian plane, this section will look at graphing the 'locus' of points that satisfies a complex equations or inequalities.

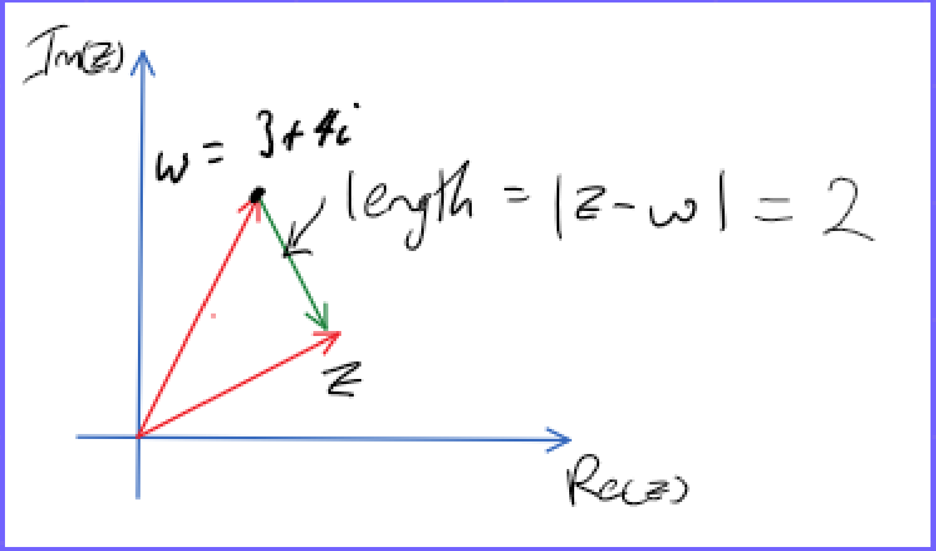


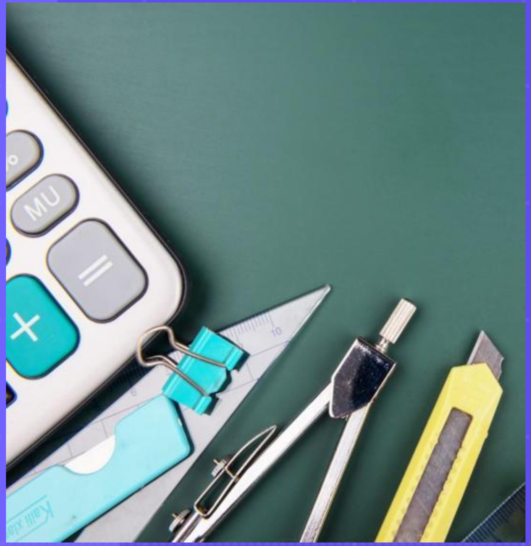
Complex plane – Type 1

Consider the following equation $|z-(3+4i)|=2$, where z is a variable complex number (e.g. a random point in the complex plane).

$|z-(3+4i)|=2$ is interpreted as the set of points where the distance between $w = 3+4i$ and some other point z, is equal to 2.

In general the locus for the equation $|z-(a+bi)|=k$ would be a circle of radius k, centred around the point (a,b). Please take note that the centre is in brackets in the equation.



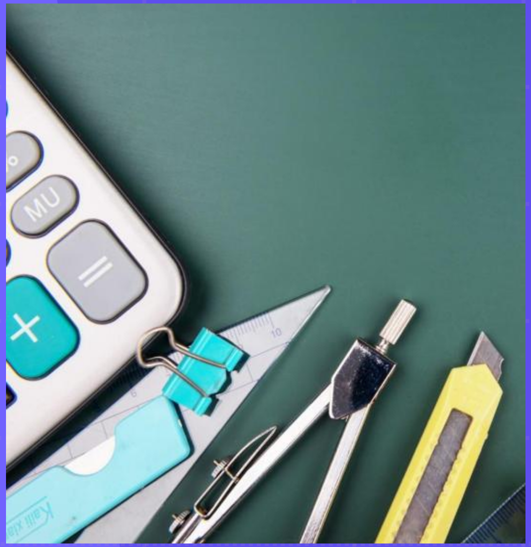


Complex plane – Type 2

Consider the following equation $|z - 2i| = |z - 4|$ where z is a variable complex number (e.g. a random point in the complex plane).

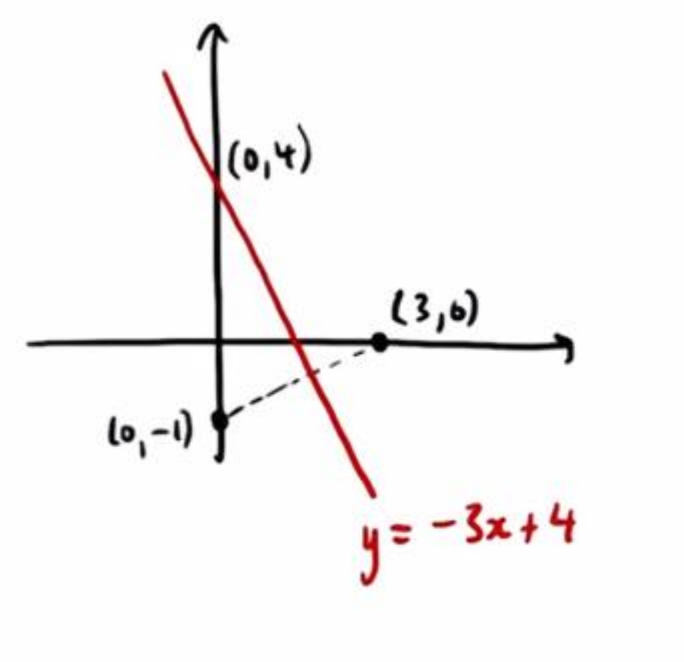
If we were to consider the locus (set of points) that satisfy this equation we could see that the two points we have are 2i and 4. These are points on the respective axes.

The question is saying to find a value z where the distance (hence use of | |) between the variable complex number z and the point 2i is the same as the distance from that same complex variable number z and the point 4.

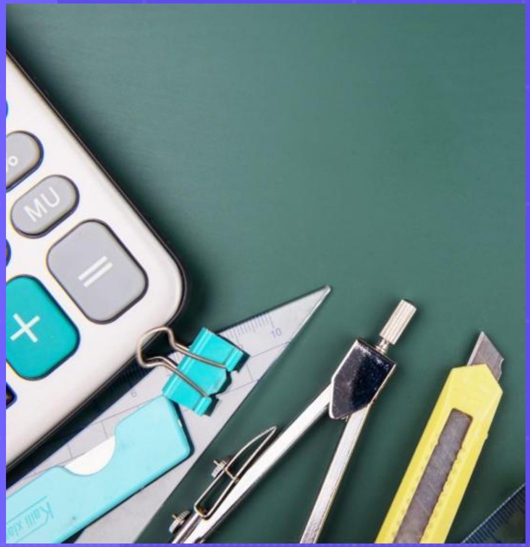


Complex plane – Type 2

Follow these steps to plot an equation such as $|z-3| = |z+\mathrm{i}|$



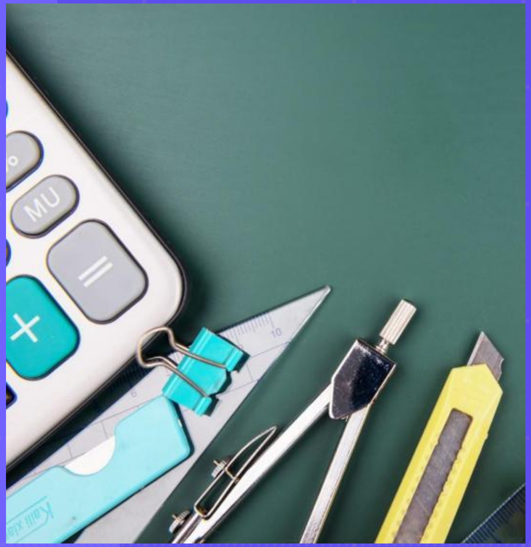
- Sketch out a set of axes (usually given in exams etc).
- Plot the points 2i (3,0) and 4 (0,-1).
- Find the point equidistance along the line connecting the two points (dashed black line).
- Draw the line which is perpendicular to connecting line (red line).
- The equation of the line will be the infinite set of values which is the solution to the equation.



Complex plane – Type 3

For the complex number $z = x + yi$, it will have the argument $\tan\theta = \frac{y}{x}$. By rearranging this equation, we can write the argument as $y = x\tan\theta$, you may remember from Year 11 Methods that this is drawn as a line with inclination θ.

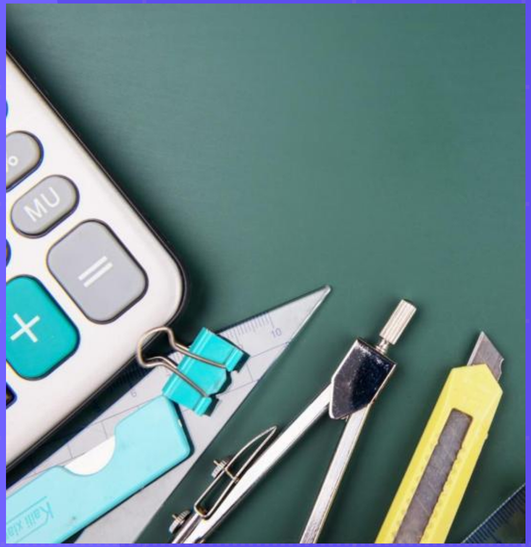
Hence if we are asked to draw the locus in the form $\arg(z) = \theta$, we will need to draw a half line that is inclined at an angle θ to the real axis.



Roots of a complex number

The fundamental theorem of algebra, also called d'Alembert's theorem or the d'Alembert–Gauss theorem, states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This includes polynomials with real coefficients, since every real number is a complex number with its imaginary part equal to zero. (Wikipedia, accessed 6/9/2026).

For example, a polynomial with degree 4, e.g. $x^4 + 3x^2$, will have four roots/solutions which may be real or complex.



Roots of a complex number

Suppose we were asked to solve the following equation:

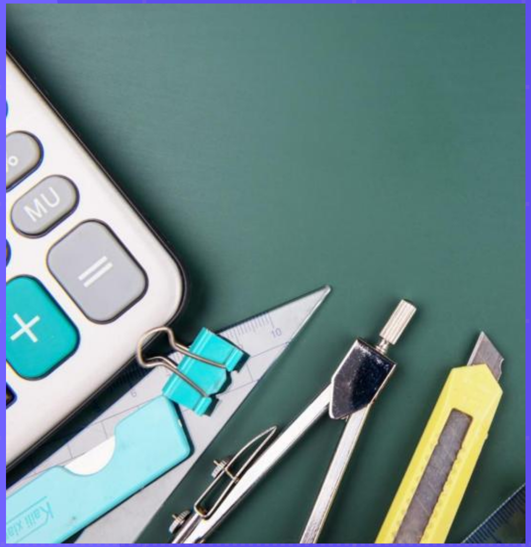
$$x^3 = 1$$

We could phrase this as being asked to find the cube roots of unity (unity meaning 1).

When attempting to solve this equation, we now know (according to the fundamental theorem of algebra) that there will be three roots.

The solution will therefore be three points, at a modulus of 1 and a spacing of $\frac{2\pi}{3}$.

As our first solution is x = 1, the other two will be complex conjugates of each other.



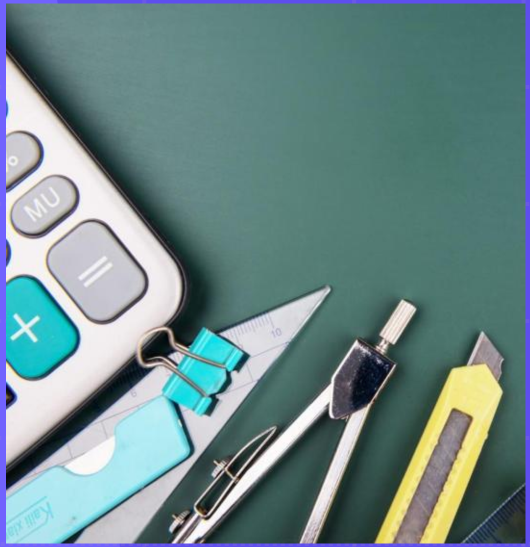
Roots of a complex number

Suppose we were asked to find the fifth roots of 1 giving exact answers in the form $r\ cis\ \theta$ with $r \geq 0$ and -$\pi < \theta \leq \pi$.

That is $$x^5 = 1$$

The solution will therefore be five points, at a modulus of 1 and a spacing of $\frac{2\pi}{5}$.

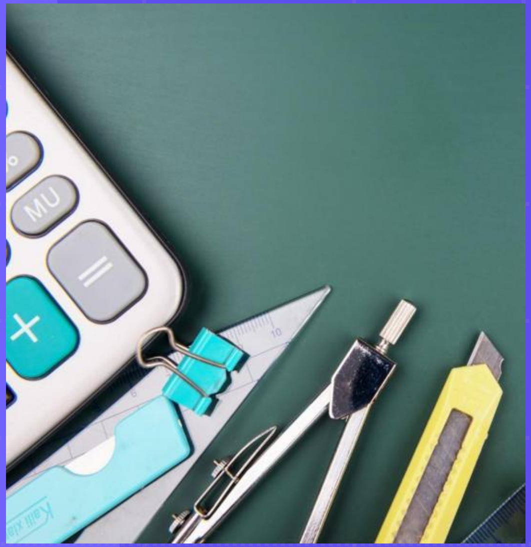
As our first solution is x = 1, the others will be two pairs of complex conjugates of each other.



Roots of a complex number

Suppose we were asked to find all six roots of $-8i$, and we were given that $(1+i)^6 = -8i$. This is telling us the first solution is at the point $(1+i)$ and the remainder will be equally spaced around the Argand plane.

Another way of writing this is $(z)^6 = -8i$, and hence z = $(1+i)$



So, our first solution is $(1+i)$ and the remainder will be spread around at a spacing of $\frac{2\pi}{6}$

Remainder theorem

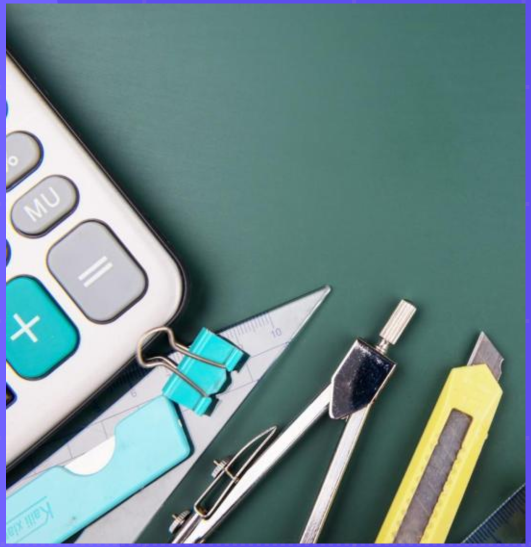
If we take a polynomial $x^4 + 3x^3 - 2x^2 + 5x - 1$ and divide it by $x - 1$, we would find the remainder to be 6.

The remainder theorem states that when the polynomial $f(x)$ is divided by $ax - b$, the remainder is given by $f\left(\frac{b}{a}\right)$. That is, the remainder is given by substituting the solution of divisor (e.g. $ax - b = 0$) into the function.

Using our example above the remainder should be given by

$$f(1) = 1^4 + 3(1)^3 - 2(1)^2 + 5(1) - 1$$

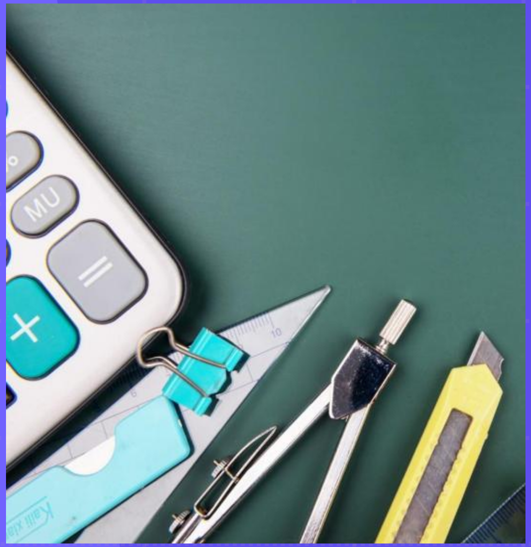
$$= 6$$

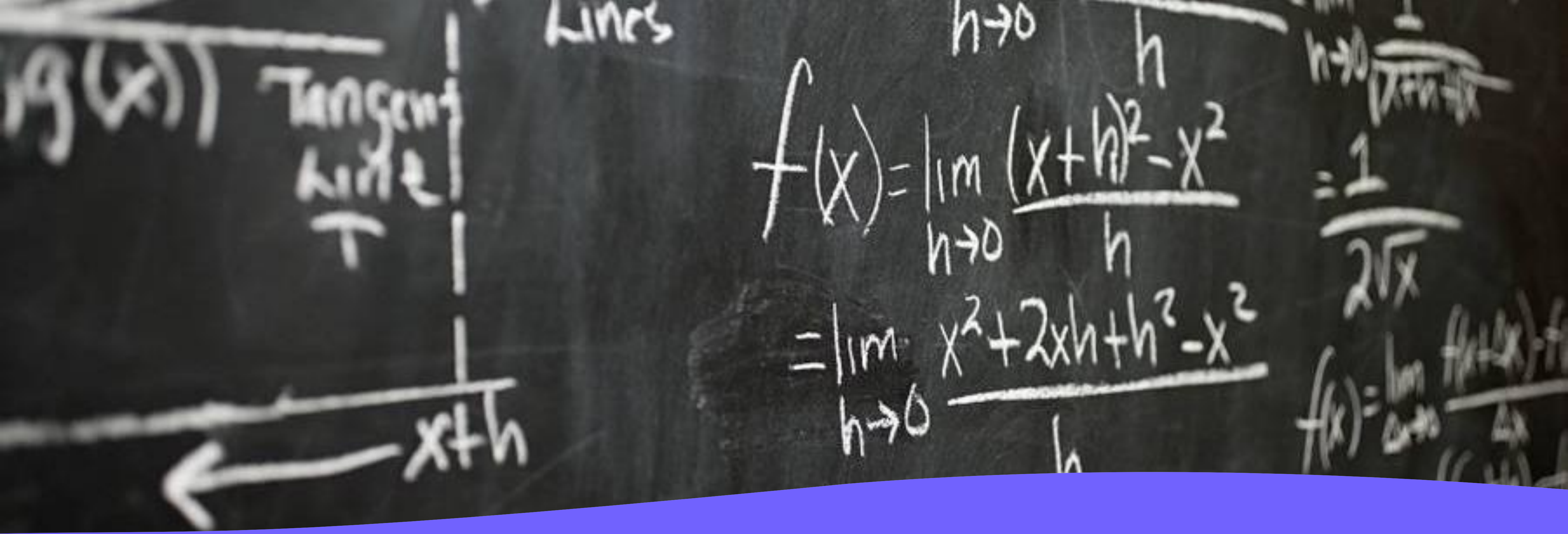


Factor theorem

If we again consider a polynomial $f(x)$ and dividing it by $(ax-b)$ leaves no remainder, then $(ax-b)$ must be a factor of $f(x)$, that is it goes in wholly (if you think of it compared to normal division). It also follows that $f\left(\frac{b}{a}\right)=0$.

$$f\left(\frac{b}{a}\right)=0 \Leftrightarrow (ax-b) \text{ is a factor of } f(x).$$



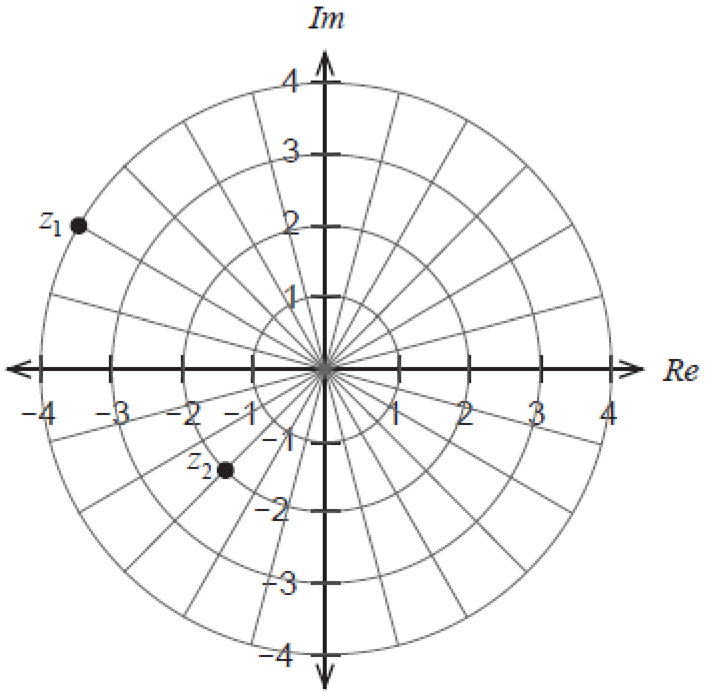

Lines
Tangent line
T
x+h
$f(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$
$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$
$= \frac{1}{2\sqrt{x}}$

Questions?

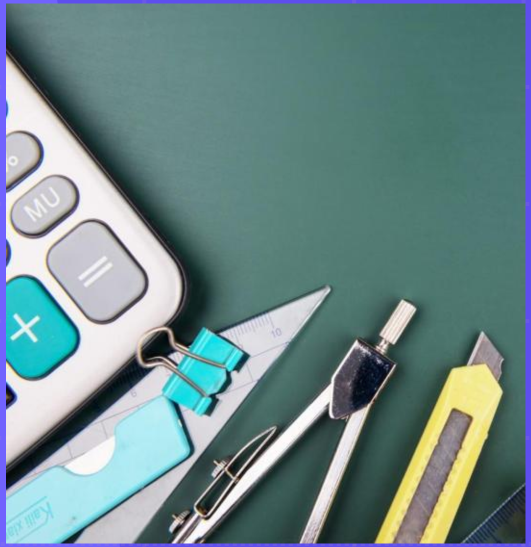
WACE 2022 Question 6 (NC)

Question 6 **(8 marks)**

Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below.



(a) Determine the exact polar form for z_2. (2 marks)

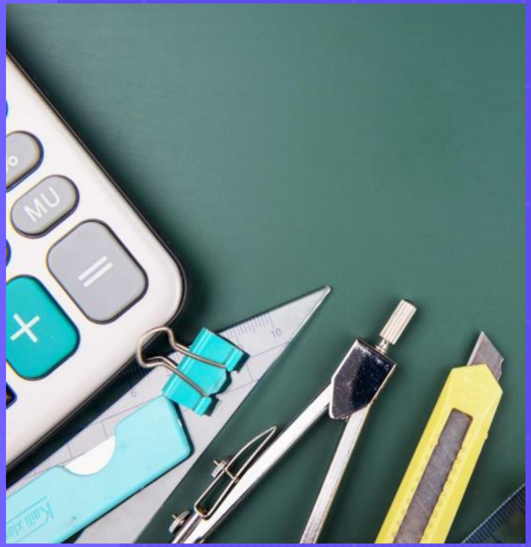


WACE 2022 Question 6 (NC)

(b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)

(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p.

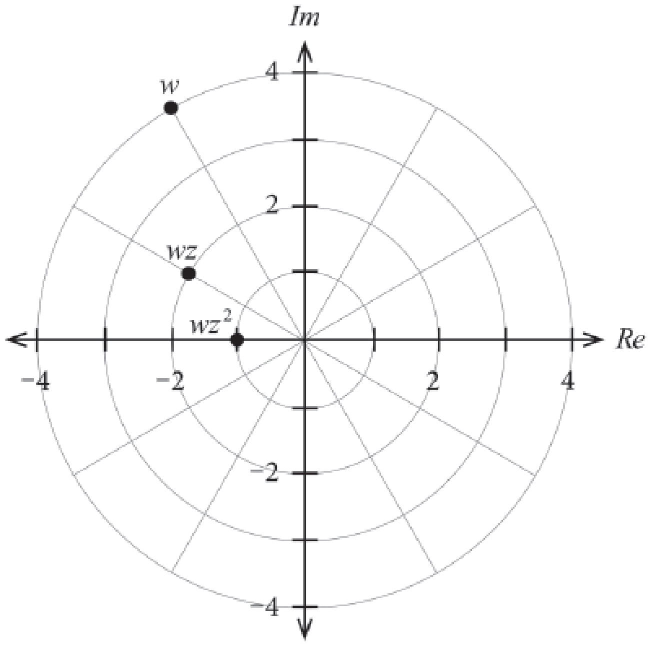
Justify your answer. (3 marks)



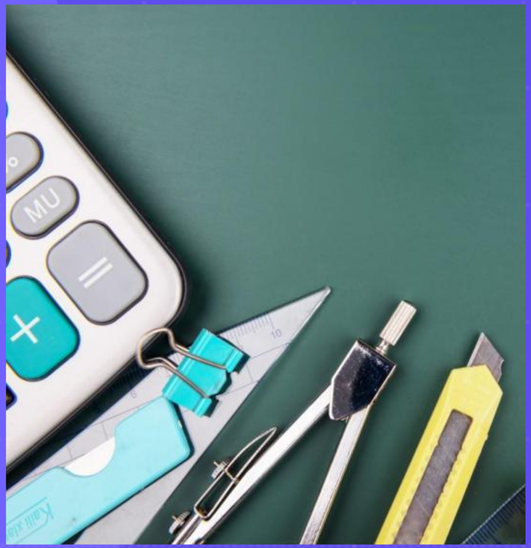
WACE 2023 Question 10 (C)

Question 10 **(7 marks)**

The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.



(a) Express wz and wz^2 in exact polar form. (2 marks)



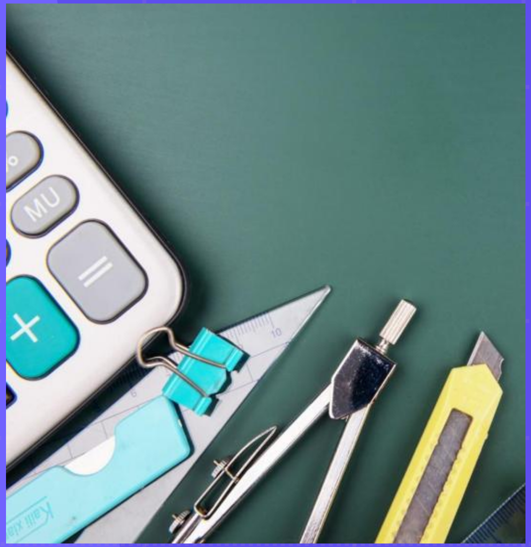
WACE 2023 Question 10 (C)

Consider the geometric transformation(s) applied to transform $w \rightarrow wz \rightarrow wz^2$ etc.

(b) Describe the geometric transformation(s) performed by successive multiplication by z.
(2 marks)

(c) Determine z in exact polar form. (1 mark)

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}.
(2 marks)



Unit 3 Topic 2

Functions and graph sketching

3.2.1 – 3.2.8

Syllabus points

Functions

3.2.1 determine when the composition of two functions is defined

3.2.2 determine the composition of two functions

3.2.3. determine if a function is one-to-one

3.2.4 find the inverse function of a one-to-one function

3.2.5 examine the reflection property of the graphs of a function and its inverse

Syllabus points

Sketching graphs

3.2.6 use and apply x for the absolute value of the real number x and the graph of y = |x|

3.2.7 examine the relationship between the graph of y = f(x) and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$

3.2.8 sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree

Combining functions

We can use the four basic operations to combine functions. That is if we take two functions; $f(x) = x^2$ and $g(x) = x - 1$.

Summing the two functions

$$f(x) + g(x) = x^2 + x - 1$$

Subtracting the two functions

$$f(x) - g(x) = x^2 - x + 1$$

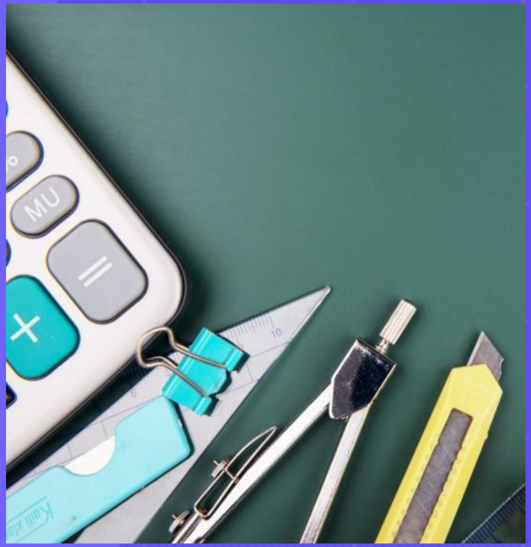
Finding the product of the two functions

$$f(x).\,g(x) = x^2(x - 1)$$

$$= x^3 - x^2$$

Dividing the two functions

$$\frac{f(x)}{g(x)} = \frac{x^2}{x-1}$$



Composite functions

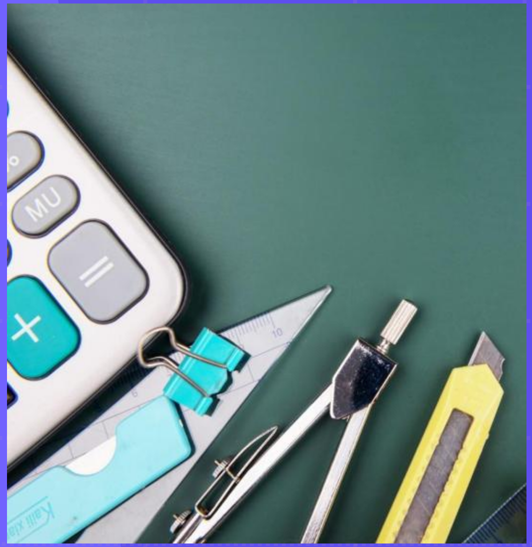
If we take a function and put it into another function, we have a composition of functions or a composite function.

Consider the two functions, $f(x) = x + 1$ and $g(x) = x^2$.

If we restrict the domain of $f(x)$ to be the numbers $x = -2, -1, 0, 1$. The range of $f(x)$ becomes R: $\{y \in \mathbb{R} |\ y = -1, 0, 1, 2\}$

If we then take this range of $f(x)$ and use it as the domain of $g(x)$ then the range of $g(x)$ becomes $\{y \in \mathbb{R} |\ y = 0, 1, 4\}$.

The range of $g\ o\ f(x)$ or $g[f(x)]$ or $gf(x)$ or $g \circ f(x)$ is $\{\mathrm{f(g(x))} \in \mathbb{R} |\ y = 0, 1, 4\}$

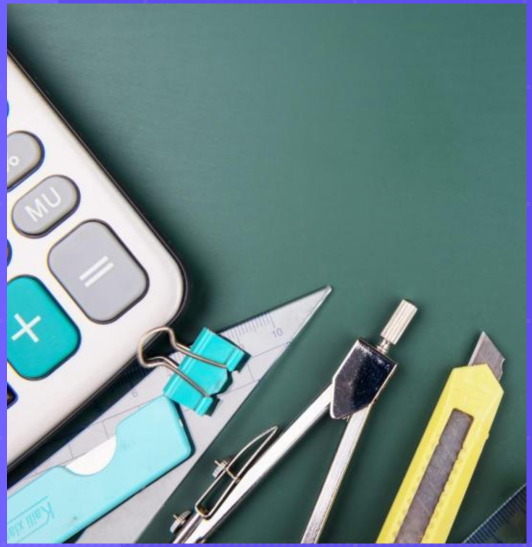


Inverse of a functions

To form an inverse relationship, we take the x and y variables and reverse their placement in the equation.

That is, if $y = 2x + 1$ then the inverse relationship is $x = 2y + 1$ or rearranged $y = \frac{x-1}{2}$

As we have this inverse relationship between a function and its inverse, we may apply this same thinking to the domain and range of a function and its inverse.



The range of f^{-1} = the domain of f

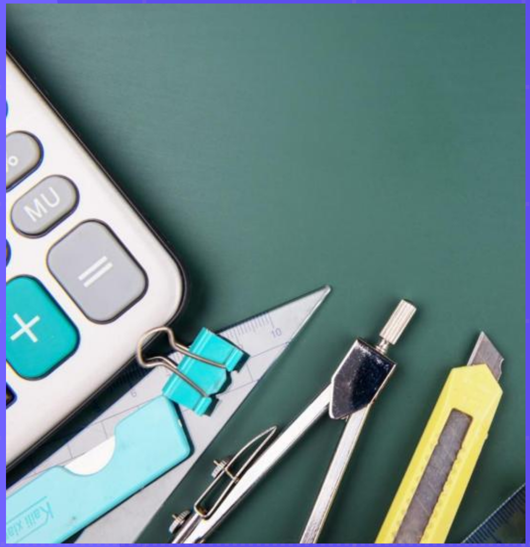
The domain of f^{-1} = the range for *f*

Inverse of a functions

But for this for this to occur, the original function needs to be a **one-to-one** function.

The inverse will be a relation if the original function is many-to-one.

Hence if we want to make a many-to-one function have an inverse that is a function and not a relation, we must restrict the original function's domain.

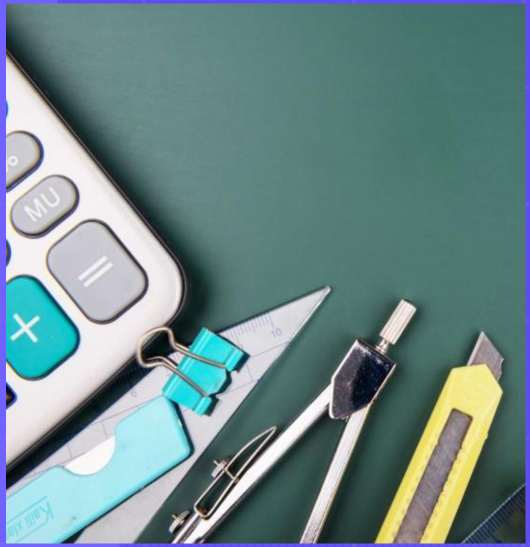


Inverse of a functions

If we consider the function $f(x) = 2x + 1$ and its inverse $f^{-1}(x) = \frac{x-1}{2}$ we may discover that the point (2, 5) from $f(x)$, there must exists a point (5, 2) on $f^{-1}(x)$.

Generally for every point (x, y) that exists on the graph of $\mathrm{f}(\mathrm{x})$, there will exist a corresponding point (y, x) on $\mathrm{f}^{-1}(\mathrm{x})$.

$\boldsymbol{y = f(x)}$	$\boldsymbol{y = f^{-1}(x)}$
Root at $x = a$	Vertical intercept at $y = a$
Vertical intercept at $y = b$	Root at $x = b$
Horizontal asymptote at $y = c$	Vertical asymptote at $x = c$
Vertical asymptote at $x = d$	Horizontal asymptote at $y = d$



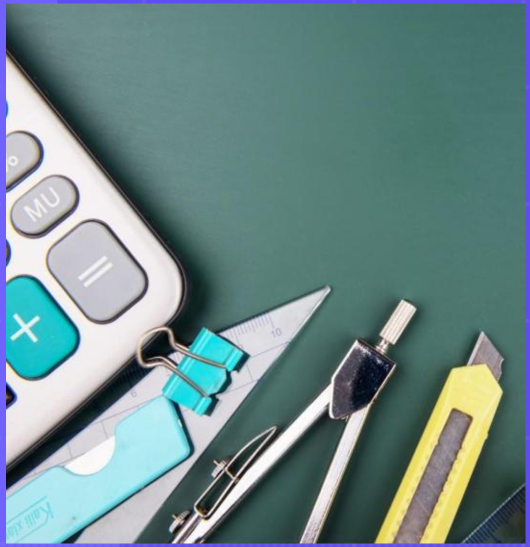
Absolute functions

The absolute value of a number is the magnitude of that number, that is, it is the distance that number is from zero.

For example the absolute value of -5, denoted as $|-5| = 5$.

In general when finding the absolute value of x, where $x < 0$, we use the values of where $x > 0$ instead. This is very similiar to finding the magnitude or modulus of a vector or complex number respectively.

Absolute value functions are particularly useful in Physics as the |velocity| = speed and |displacement| = distance

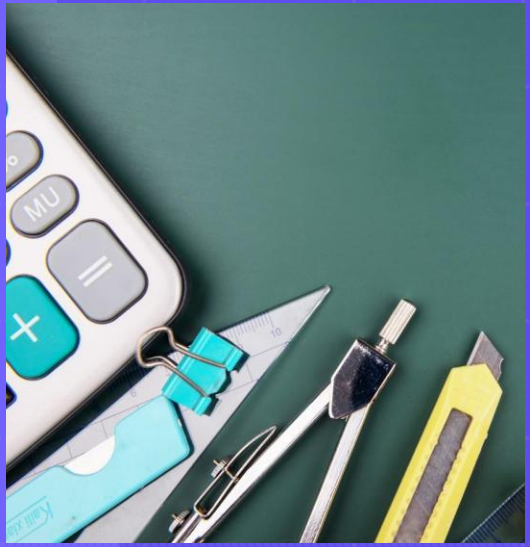


Absolute functions

Our formula sheet gives us (the quadratic formula doesn't really fit anywhere, but it might be useful somewhere!)

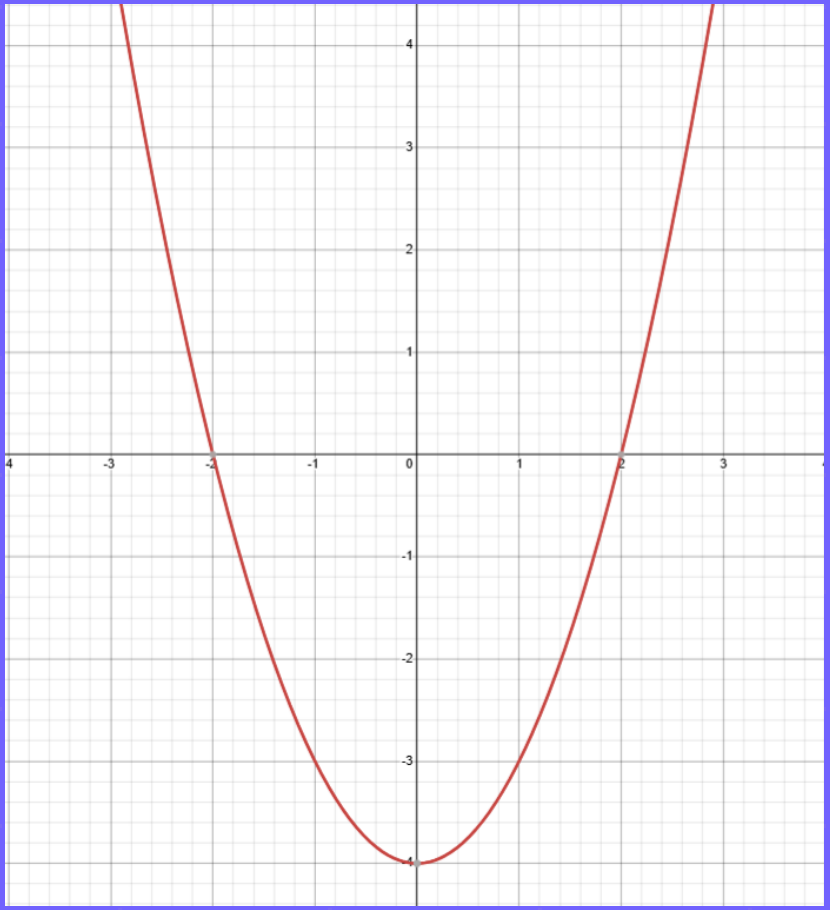
Functions

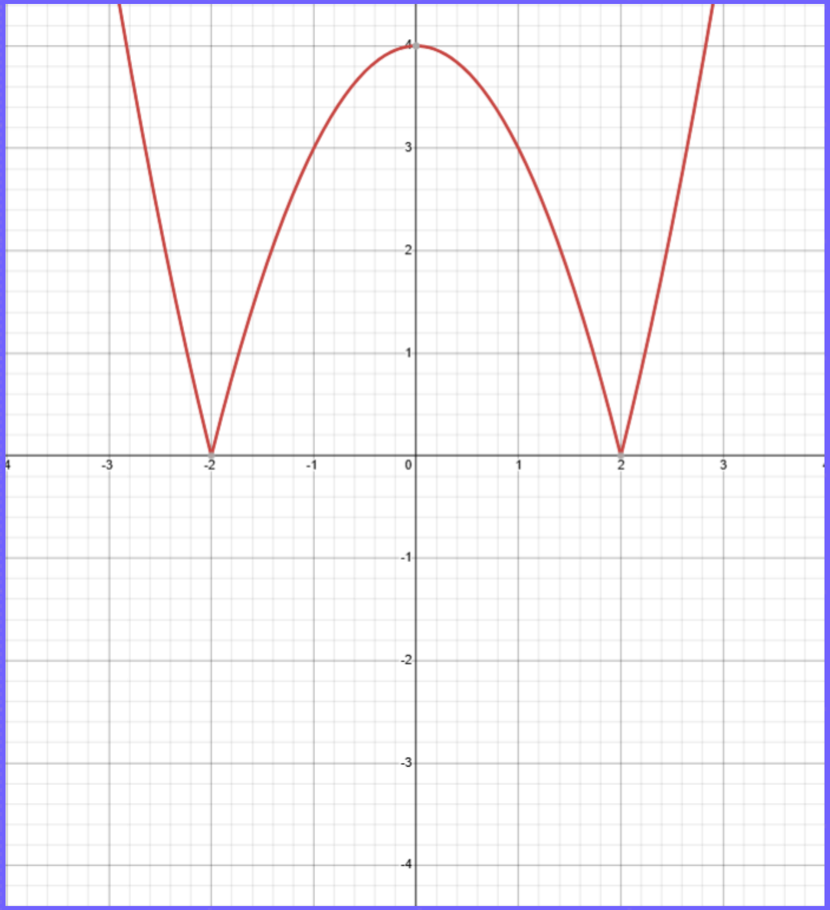
Quadratic function	If $f(x) = ax^2 + bx + c$ and $f(x) = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Absolute value function	$\lvert x \rvert = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$

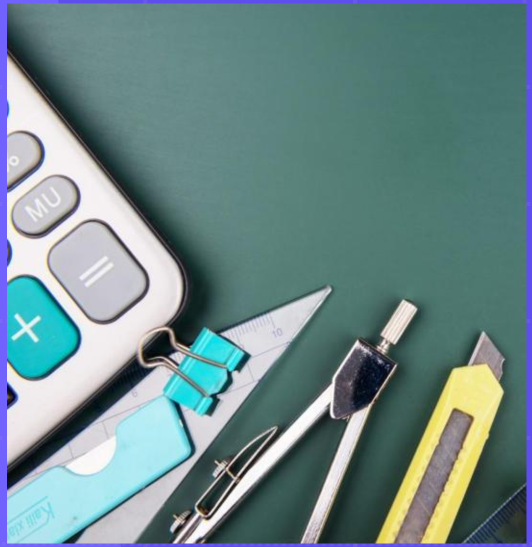


Graphing $y = |f(x)|$

Consider the following function $f(x) = x^2 - 4$ and $y = |f(x)|$

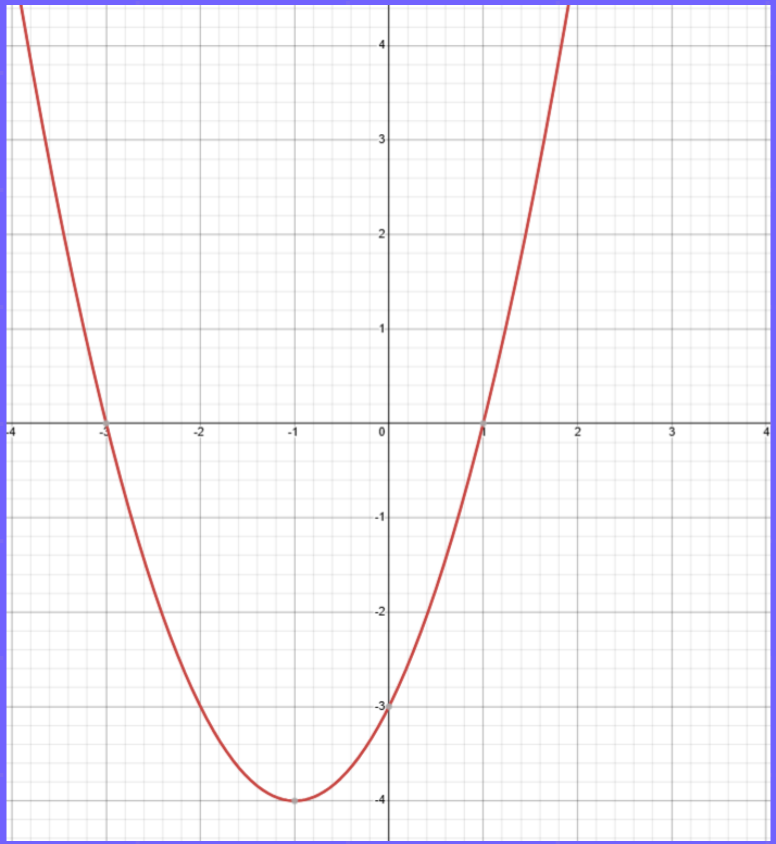


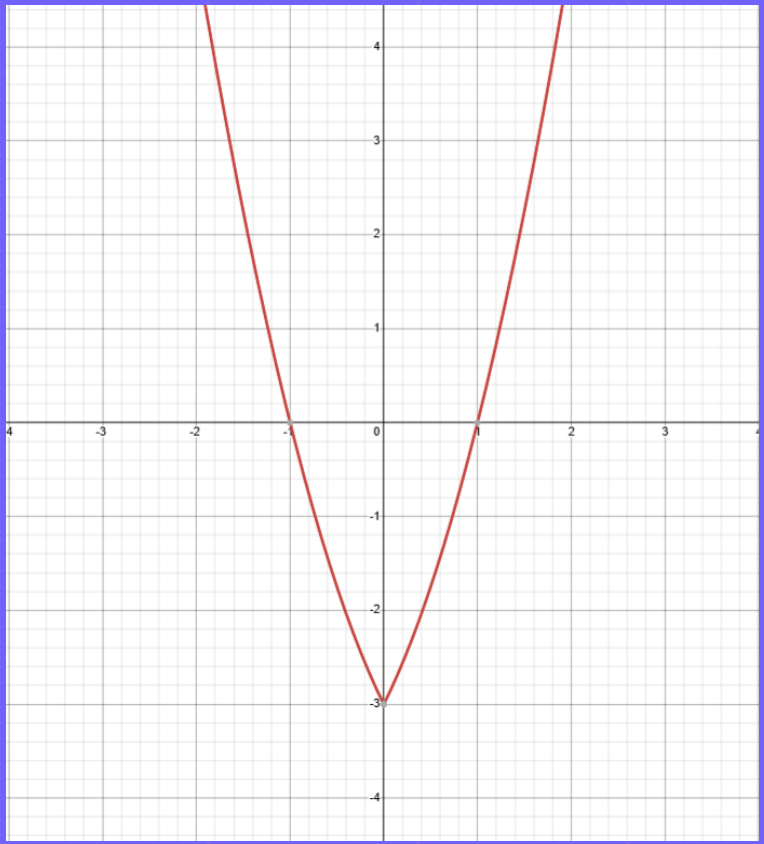


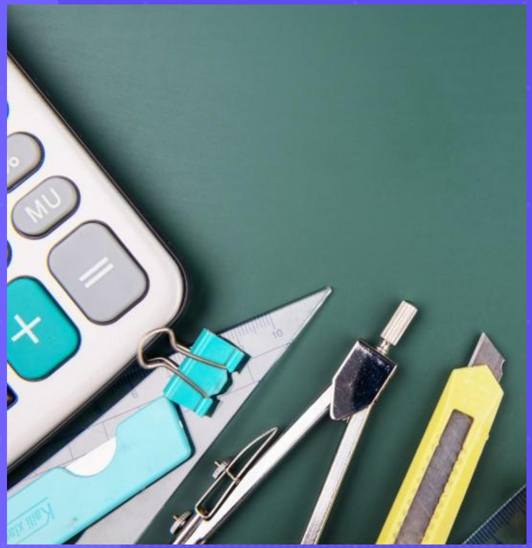


Graphing $y = f(|x|)$

Consider the following function $y = f(|x|)$, where $f(x) = (x-1)(x+3)$ and $y = (|x|-1)(|x|+3)$



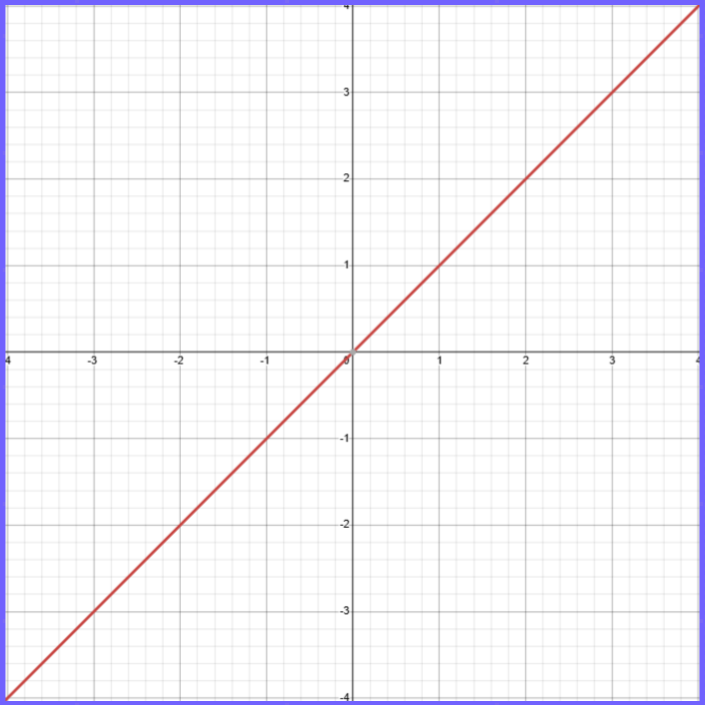


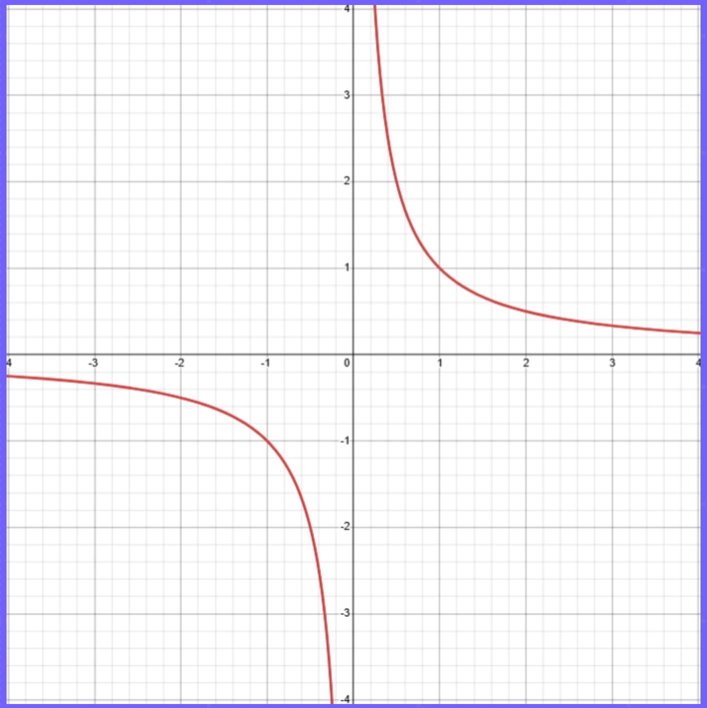


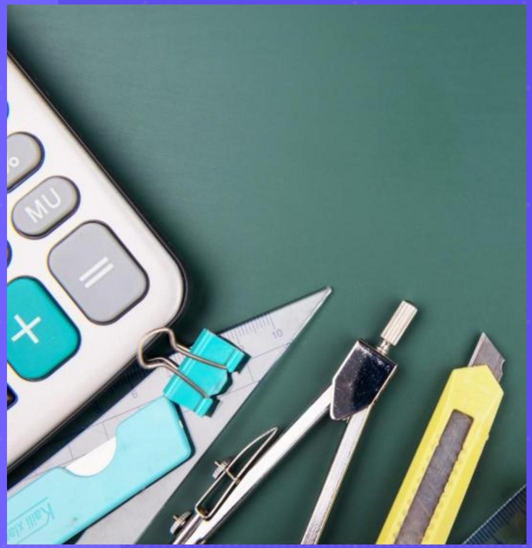
Graphing $y = \frac{1}{f(x)}$

Called the reciprocal function (not to be confused with the inverse)

If we think of the graph $y = x$, then the hyperbola $y = \frac{1}{x}$ is its reciprocal!



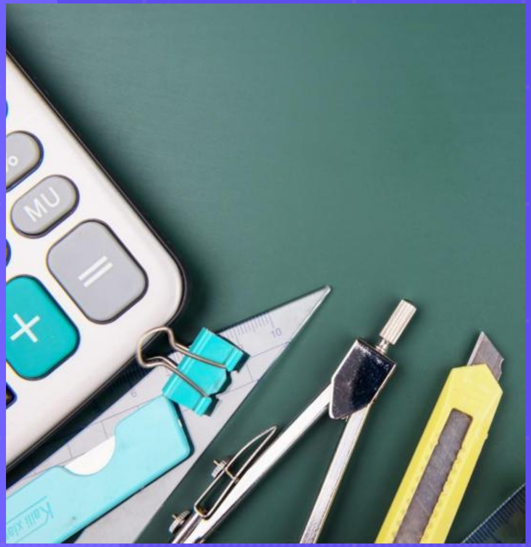




Graphing $y = \dfrac{1}{f(x)}$

Given the graph of $y = f(x)$, if we attempt to graph $y = \frac{1}{f(x)}$ we can approach this by thinking how the y-coordinates change between these two graphs. For every point (x, y) on the graph of $f(x)$, there corresponds a point on $\frac{1}{f(x)}$ of $\left(x, \frac{1}{y}\right)$. That is the x-coordinates do not change, but the y-coordinates become $\frac{1}{y}$.

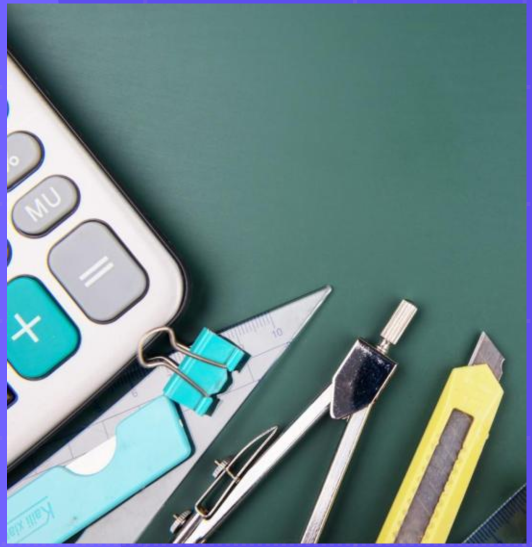
$\boldsymbol{y = f(x)}$	$\boldsymbol{y = \dfrac{1}{f(x)}}$
x-intercept at $(c, 0)$	Vertical asymptote at $x = c$
y-intercept at $y = b$	y-intercept at $y = \frac{1}{b}$
Horizontal asymptote at $y = p$	Horizontal asymptote at $y = \frac{1}{p}$
Vertical asymptote at $x = k$	x-intercept at $(k, 0)$
Local minimum at (x, y)	Local maximum at $\left(x, \frac{1}{y}\right)$
Local maximum at (x, y)	Local minimum at $\left(x, \frac{1}{y}\right)$



Graphing rational functions

A rational function is one where $y = \frac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ are polynomials. These types of functions will have horizontal and vertical asymptotes as you would expect but may also have oblique asymptotes for which we will have to use algebraic long division.

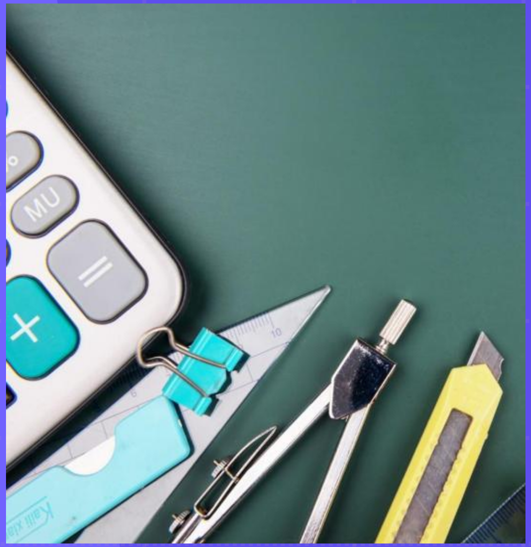
3.2.8 sketch the graphs of simple rational functions where the numerator and denominator are polynomials of **low degree ???**



Graphing rational functions

There are two types of rational functions that we will consider in this course:

- Proper rational functions - this is where the polynomial in the numerator is of a lesser degree than the denominator (much like a proper fraction).
- Improper rational functions - this is where the polynomial in the numerator is of the same or greater degree than the denominator (much like an improper fraction). When faced with an improper rational function we will use algebraic long division to break it down to identify its key features more easily.



To 'see' the difference, remove 'rational function' and replace it with 'fraction'

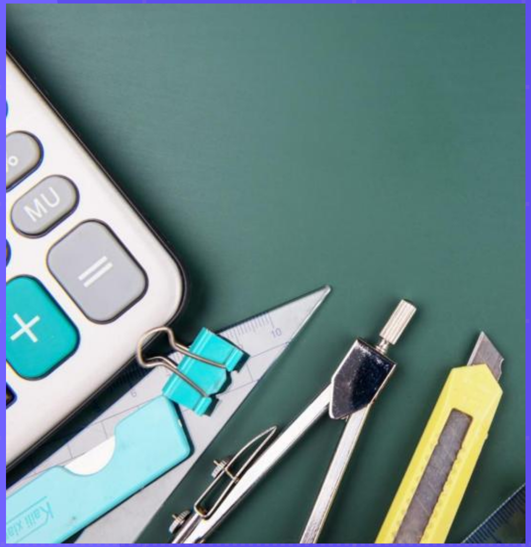
Proper fraction – numerator is large than the denominator Eg $\frac{9}{5}$

Improper fraction – numerator is smaller than the denominator Eg $\frac{2}{3}$

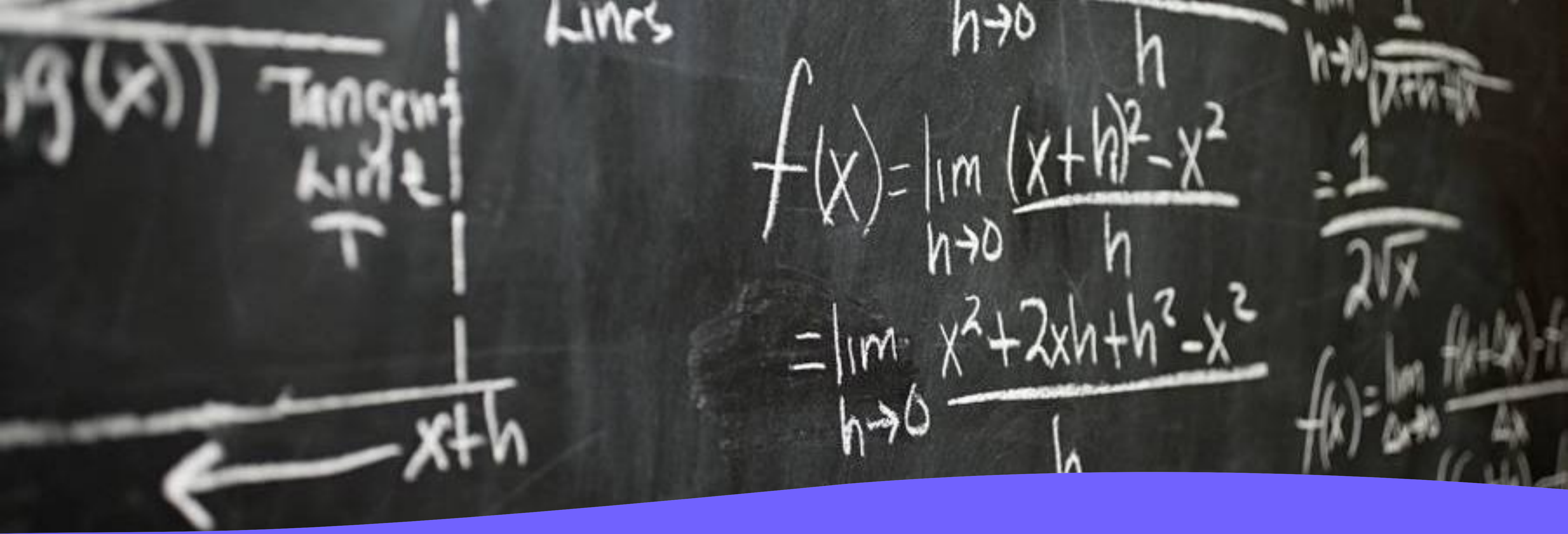
Key features of rational functions

When sketching rational functions, we usually start by sketching the key features of the graph. These are:

- x- and y-intercepts. Found by letting $y = 0$ for x-intercepts and letting $x = 0$ for y-intercepts.
- Behaviour as $x \to \infty$ and $x \to -\infty$
- Asymptotes - horizontal, vertical or oblique.
- 'Holes' or their proper term: points of discontinuity in the graph. This occurs if the numerator and denominator have a linear factor in common (e.g. both the numerator and denominator have a factor of $(x + 3)$).
- Turning points. Usually found using calculus.



There is a nice example to test yourself on this idea coming up.

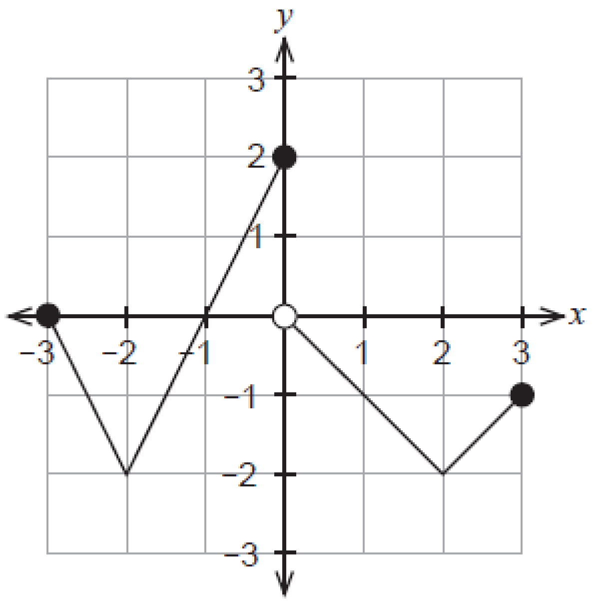


Questions?

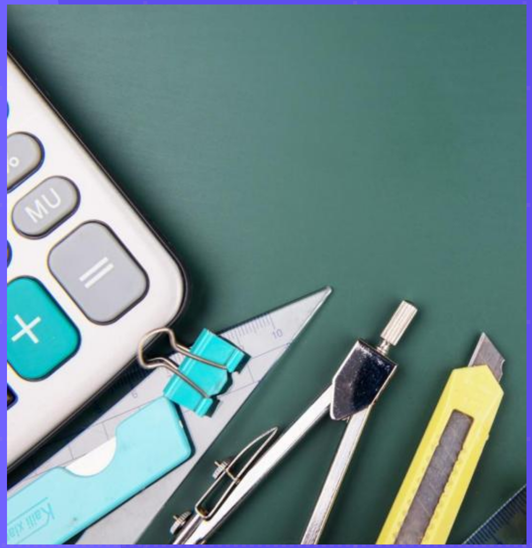
WACE 2022 Question 2 (NC)

Question 2 **(7 marks)**

The graph of $y = f(x)$ is shown below.

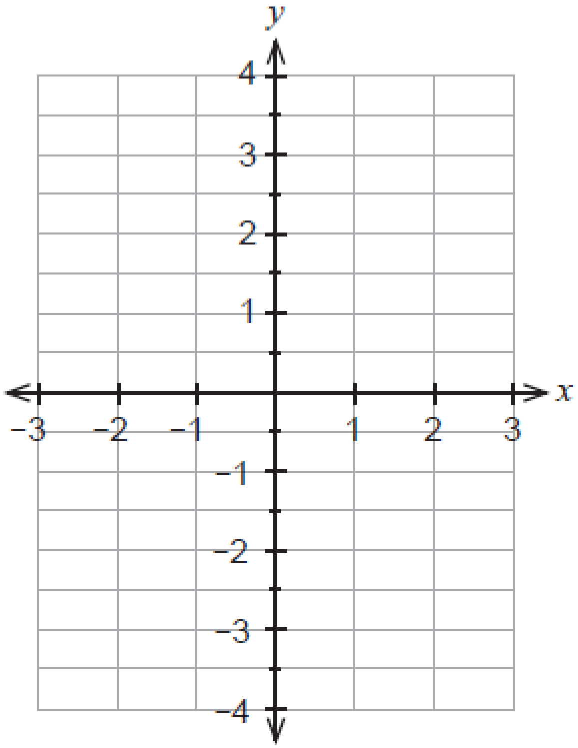


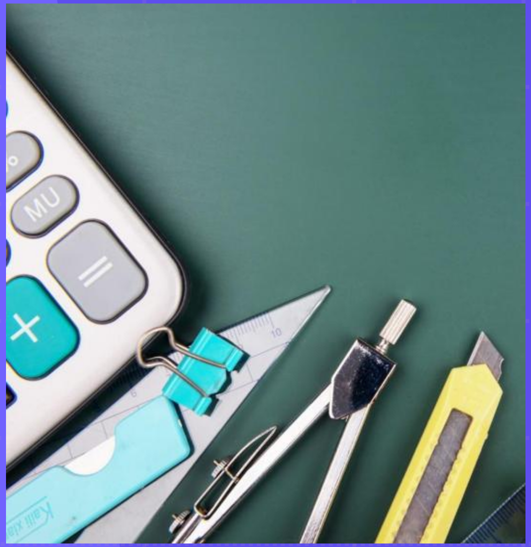
(a) Solve the equation $|f(x)| = x$. (2 marks)



WACE 2022 Question 2 (NC)

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)



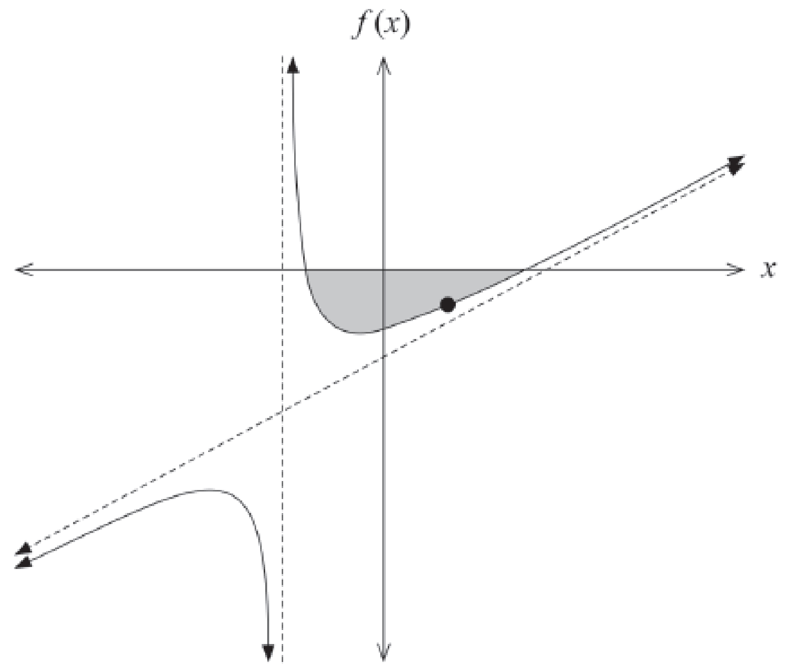


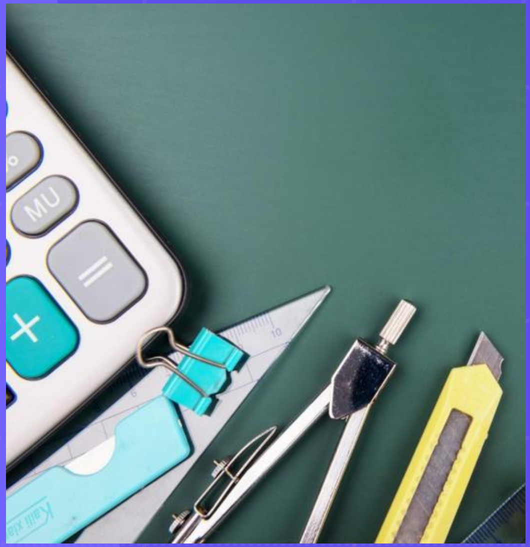
WACE 2023 Question 18 (C)

Question 18 **(6 marks)**

Function $f(x)$ is a rational function of the form $\dfrac{x^2+bx+c}{x+d}$ with the following properties:

- $f(2)=-2$
- $f(x)$ has a vertical asymptote at $x=-3$ and another asymptote with equation $y=x-5$.

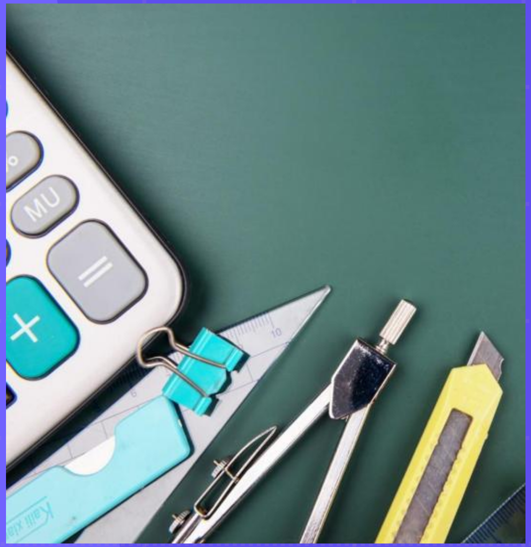


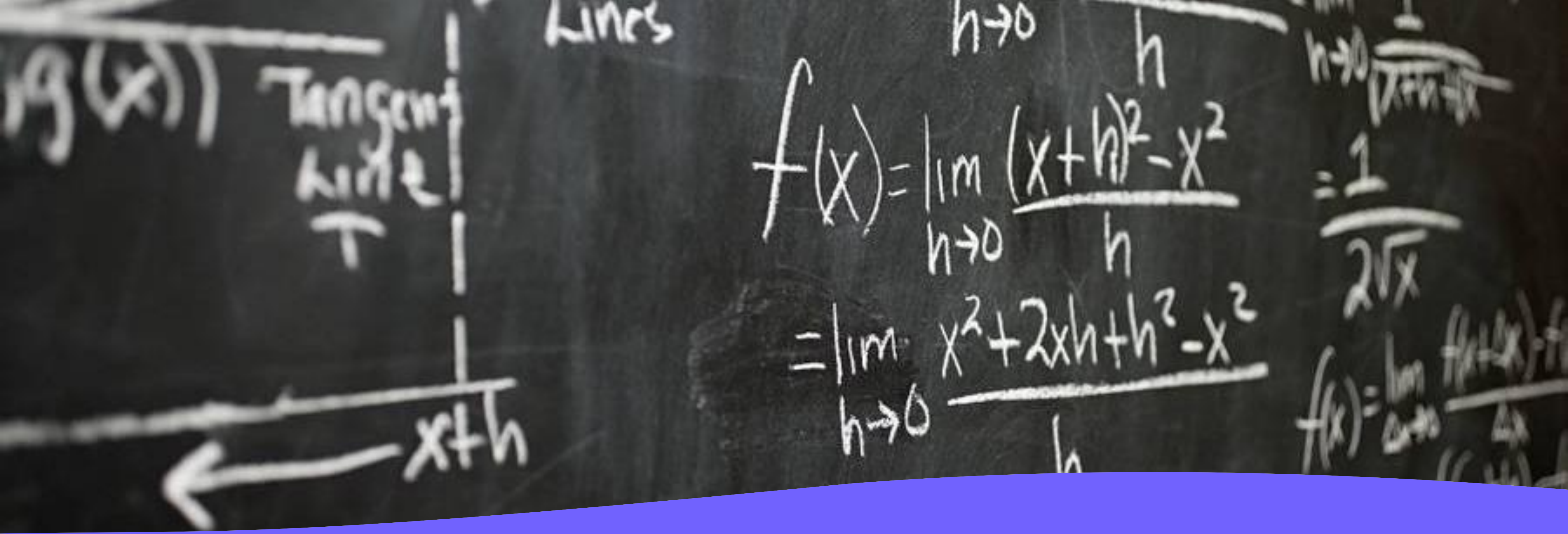


WACE 2023 Question 18 (C)

(a) Show that $b = -2$, $c = -10$ and $d = 3$. (3 marks)

(b) Calculate the exact area of the shaded region. (3 marks)





Time to take a break!

10 minutes to stretch your legs, have a chat, use the restrooms!

Unit 3 Topic 3

Vectors in three dimensions

3.3.1 – 3.2.15

Syllabus points

The algebra of vectors in three dimensions

3.3.1 define the concept of a vector in three dimensions, using the unit vectors i, j and k, determining magnitude, scalar (dot) product and parallel and perpendicular vectors

3.3.2 prove geometric results in the plane and construct simple proofs in 3 dimensions

Syllabus points

Vector and Cartesian equations

3.3.3 introduce Cartesian coordinates for three dimensional space, including plotting points and equations of spheres

3.3.4 use vector equations of curves in two or three dimensions involving a parameter and determine an equivalent Cartesian equation in the two-dimensional case

3.3.5 determine a vector equation of a straight line and straight line segment, given the position of two points or equivalent information, in both two and three dimensions

3.3.6 examine the positions of two particles, each described as vector functions of time, and determine if their paths cross or if the particles meet

3.3.7 use the cross product to determine a vector normal to a given plane

3.3.8 determine vector and Cartesian equations of a plane

Syllabus points

Systems of linear equations

3.3.9. recognise the general form of a system of linear equations in several variables, and use elementary techniques of elimination to solve a system of linear equations

3.3.10 examine the three cases for solutions of systems of equations – a unique solution, no solution, and infinitely many solutions – and the geometric interpretation of the solution of a system of equations with three variables

Syllabus points

Vector calculus

3.3.11 consider position vectors as a function of time

3.3.12 derive the Cartesian equation of a path given as a vector equation in two dimensions, including ellipses and hyperbolas

3.3.13 differentiate and integrate a vector function with respect to time

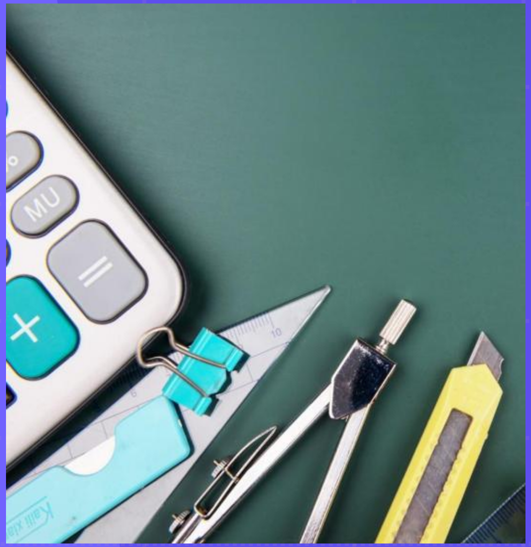
3.3.14 determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration

3.3.15 apply vector calculus to motion in a plane, including projectile and circular motion

3D Vector basics

Some basics from the formula sheet

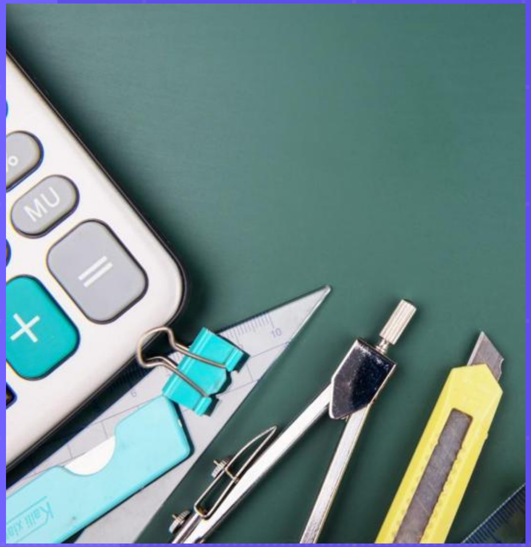
Magnitude	$\lvert(a_1, a_2, a_3)\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$
Dot product	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$
Cross product	$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$



3D Vector basics

Most simple 3D vector work is just an extension of 2D vector work. We introduce *k* to the *i* and *j* components.

Let's revise our 2D work first.



Vectors as a function of time

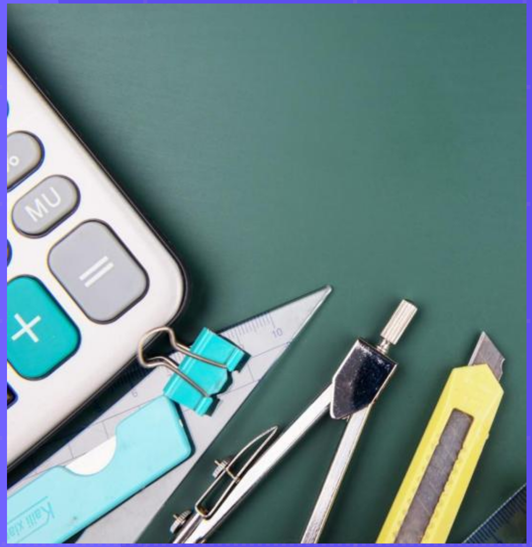
Consider that a boat leaves a dock (point A) and heads with a constant velocity (**v**) described in ms^{-1}. We can then say that the boat will be at some position vector **r** (or point R), *t* seconds later (where A and R are measured from the origin).

Hence, using vector addition:

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$$

Or

$$\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$$



Hence we can describe any particle's motion in terms of *t* and determine if two particle's paths will cross (intersect) or collide (that they are at the same position at the same time).

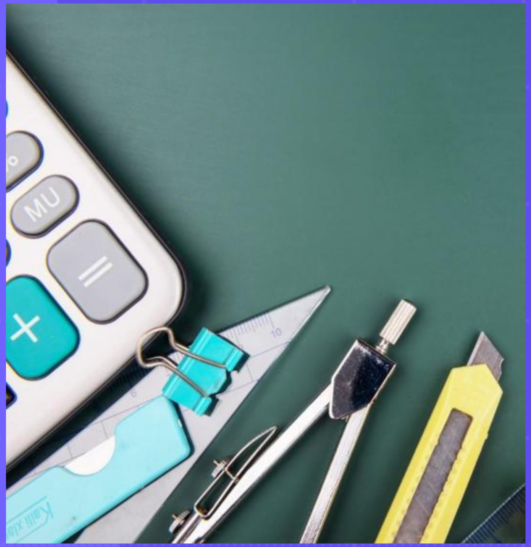
Vectors as a function of time

A question such as this can be done in 2D but in 3D the process is the same.

At noon boats P and Q have position vectors (**r**) and velocity vectors (**v**) as follows:

$\boldsymbol{r}_P = -85\boldsymbol{i}$ km $\qquad \boldsymbol{v}_P = (6\boldsymbol{i} + 8\boldsymbol{j})$ km/h

$\boldsymbol{r}_Q = (-95\boldsymbol{i} + 40\boldsymbol{j})$ km $\qquad \boldsymbol{v}_Q = (10\boldsymbol{i} - 8\boldsymbol{j})$ km/h



Hence

$$p(t) = -85i + (6\boldsymbol{i} + 8\boldsymbol{j})t \qquad q(t) = (-95\boldsymbol{i} + 40\boldsymbol{j}) + (10\boldsymbol{i} - 8\boldsymbol{j})t$$

Vectors equation of a line

Given the four colinear points A, B, C and D an equation to describe the line could be written as

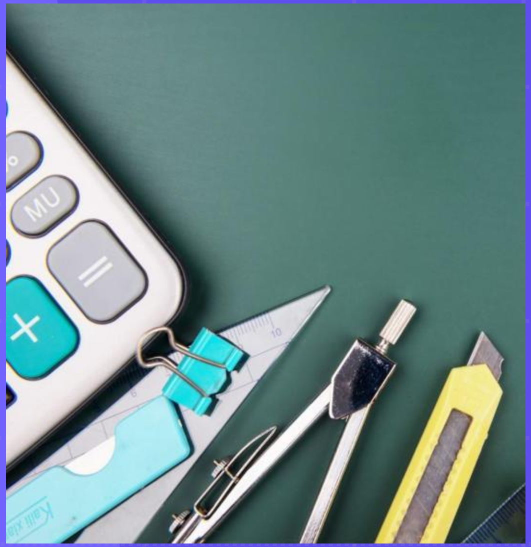
$$r \ = \ \overrightarrow{OA} + \ \lambda\overrightarrow{AB}$$

Or more generally

$$\boldsymbol{r} = \boldsymbol{a} + b\lambda$$

Where *a* and *b* are points on the line.

This is extremely similar (not surprisingly!) to $y = \mathrm{m}x + c$



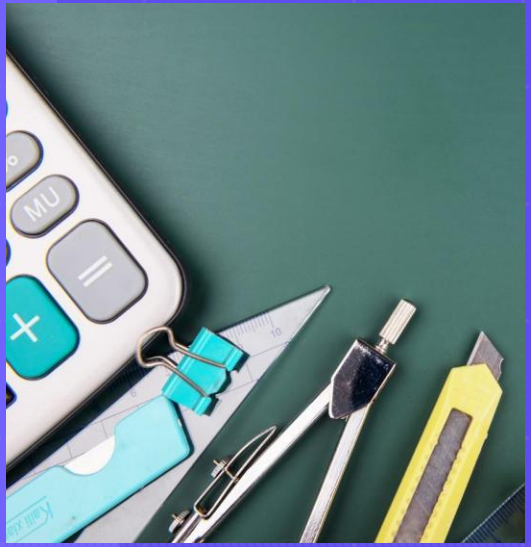
Vectors equation of a line

Our formula sheet gives us

Equation of a line	One point and direction	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$
	Two points A and B	$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

Cartesian equation of a line	$\dfrac{x-a_1}{u_1} = \dfrac{y-a_2}{u_2} = \dfrac{z-a_3}{u_3}$

Parametric equation of a line	$x = a_1 + \lambda u_1 \ldots \ldots (1)$ $y = a_2 + \lambda u_2 \ldots \ldots (2)$ $z = a_3 + \lambda u_3 \ldots \ldots (3)$



Vectors to Cartesian equation

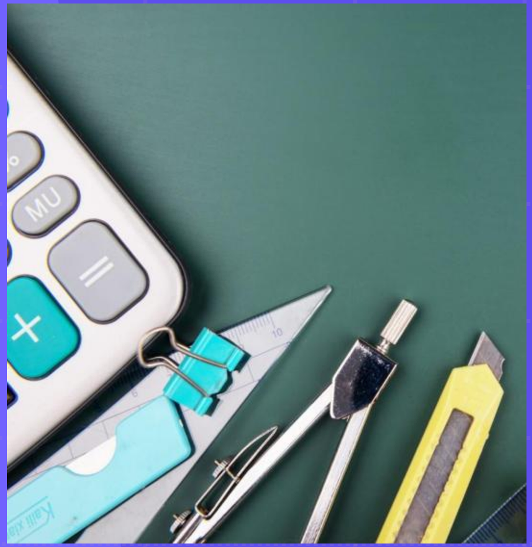
Consider the line $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(-\boldsymbol{i} - 5\boldsymbol{j})$ we should be able to convert this to a Cartesian equation.

Consider the equations

$$x = 3 - \lambda$$
$$y = 2 - 5\lambda$$

We can then use simultaneous equations to eliminate λ and express our answer y in terms of x.

It is important to note that all three forms (vector, parametric and Cartesian form) is describing the exact same line, and we should expect that each form would produce the same results.



Vectors to Cartesian equation

Consider the line $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(-\boldsymbol{i} - 5\boldsymbol{j})$ we should be able to convert this to a Cartesian equation via parametric equations

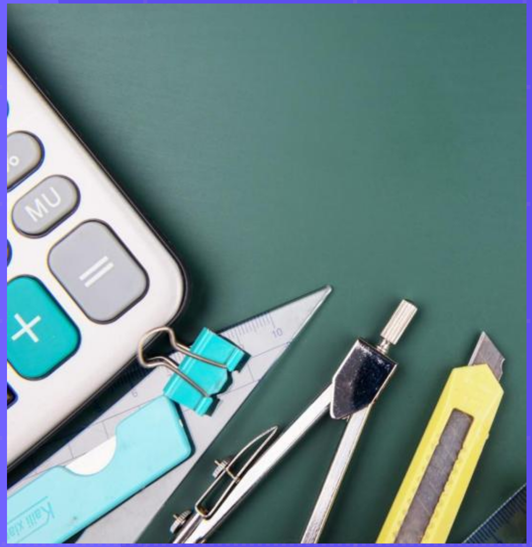
Consider the parametric equations

$$x = 3 - \lambda$$
$$y = 2 - 5\lambda$$

We can then use simultaneous equations to eliminate λ and express our answer y in terms of x.

$$y = 5x \; - 13$$

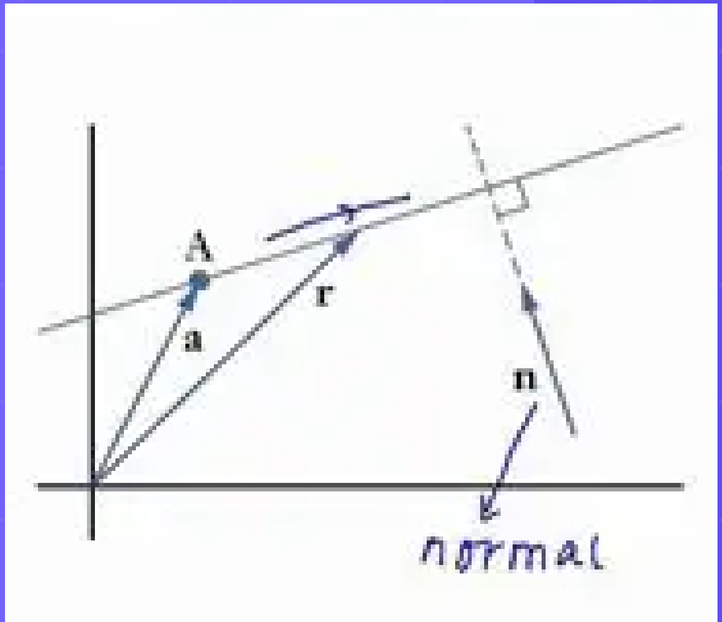
It is important to note that all three forms (vector, parametric and Cartesian form) is describing the exact same line, and we should expect that each form would produce the same results.



Scalar product form of equations

Lastly, we need to consider the scalar product form of a line.

Consider	$\underline{r} - \underline{a}$
Multiply by norma and equate to 0l	$(\underline{r} - \underline{a}).\underline{n} = 0$
Expand brackets	$\underline{r}.\underline{n} - \underline{a}.\underline{n} = 0$
Rearrange	$\underline{r}.\underline{n} = \underline{a}.\underline{n}$
Let $\underline{a}.\underline{n} = c$	$\underline{r}.\underline{n} = c$



This is the scalar product form of a line.

Vectors equations of curves

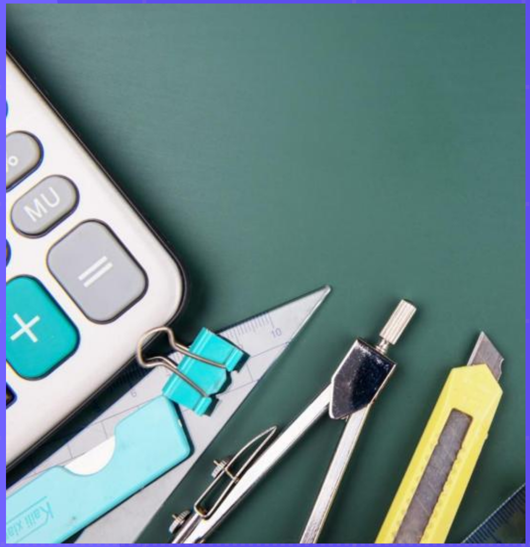
It should make sense that if we can convert lines from Cartesian form to vector form, then we should also be able to represent curves in vector form too.

Consider the equation

$$r = (t + 2)i + 3t^2 j$$

Using a similar process via parametrically written equations we can conclude

$$y = 3(x - 2)^2$$



Vectors equations of circles

Similarly, circles can be represented by a vector equation.

Firstly, consider circles at the origin and consider the equation

$$|r| = a$$

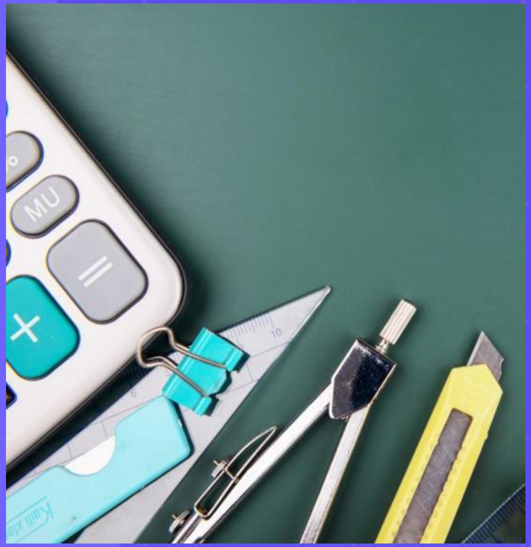
Where r is some position vector on the circle, it can be represented by

$$r = x\boldsymbol{i} + y\boldsymbol{j}$$

Hence

$$|x\boldsymbol{i} + y\boldsymbol{j}| = a$$

This can be shown to be equal to the Cartesian equation $x^2 + y^2 = a^2$ by using Pythagoras or by using sine/cos for their components.



Vectors equations of circles

Now consider circles not at the origin and consider the equation

$$|r - d| = a$$

Where r is some position vector on the circle, and d is the vector representing the movement of the centre away from the origin.

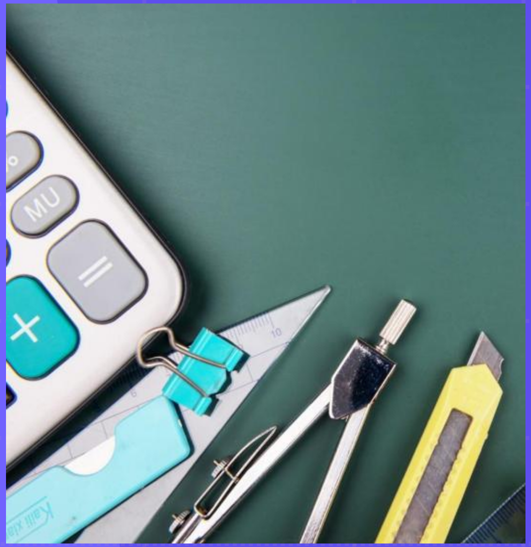
This can similarly be shown to be equal to the Cartesian equation for a circle not at the origin of $(x - p)^2 + (y - q)^2 = a^2$

From Year 11 Formula Sheet (note the co-efficients are differently named)

Graphs and Relations

Equation of a circle: $(x-a)^2 + (y-b)^2 = r^2$

where, (a,b) is the centre and r is the radius

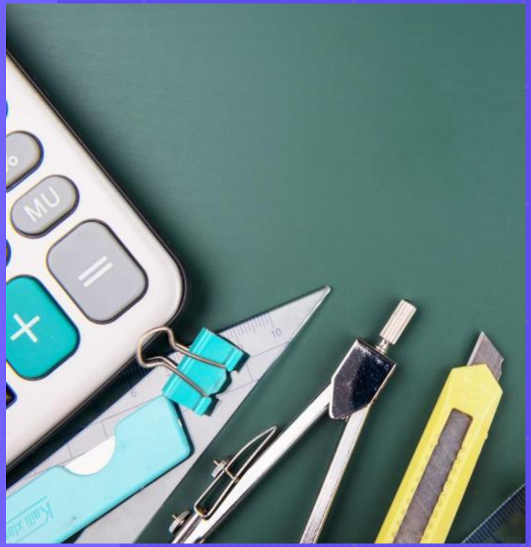


Closest approach

Where two objects are not going to collide, there is some point where they are the closest they will achieve. After that point they will forever get further apart.

Consider the situation of an oil tanker and a stranded (maybe damaged mast) small sailboat. The oil tanker can not change their course (well not easily), but they want to deploy a small rescue boat to the sailboat. Finding when to do this is closest approach.

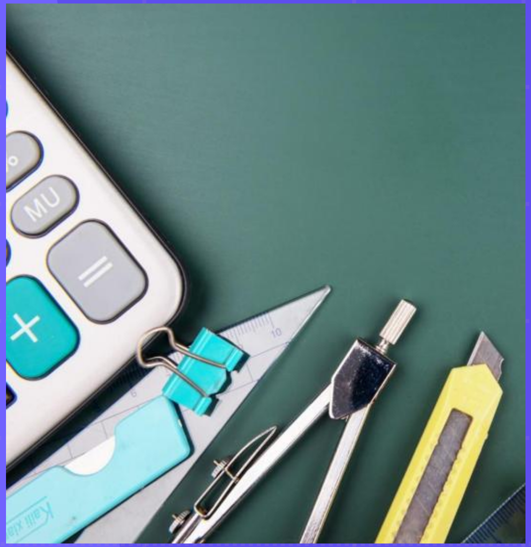
There are three methods to calculate these, using calculus, using relative velocities and trigonometry, and relative velocities and scalar product.



Closest approach – using calculus

I believe using calculus is the least cumbersome and most intuitive style to use

- Write each particle in the form of a line $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b}t$
- Determine the separation vector by $r_A - r_B$
- Determine the magnitude of the separation vectors (square the i component and the j component)
- Let $d^2 = y$ and t $= x$
- Use calculus (Classpad) to determine the minimum value
- Ensure your answer is answering the question asked!

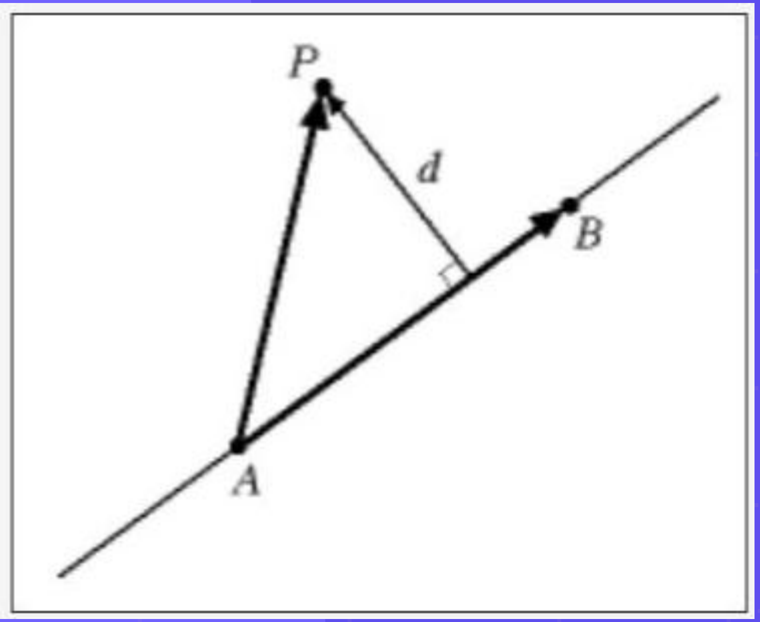


Distance from a point to a line

When asked for distance from a point to a line, we are really in a case where one particle is stationary and the other is moving.

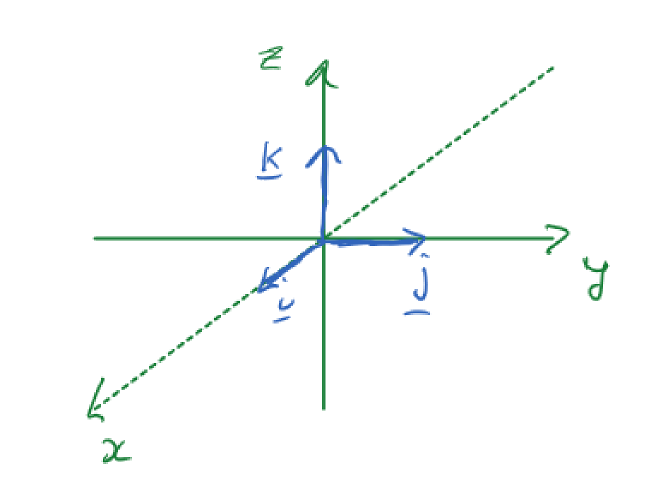
The shortest distance will a line which intersects the path of the particle at 90 degrees.

- Write the line d as –P + A + λB
- Dot this line with line AB
- Solve this for equals to 0, solving for λ
- Sub λ into the equation of the line to find the closest point
- Determine the vector for line d, and hence |d|

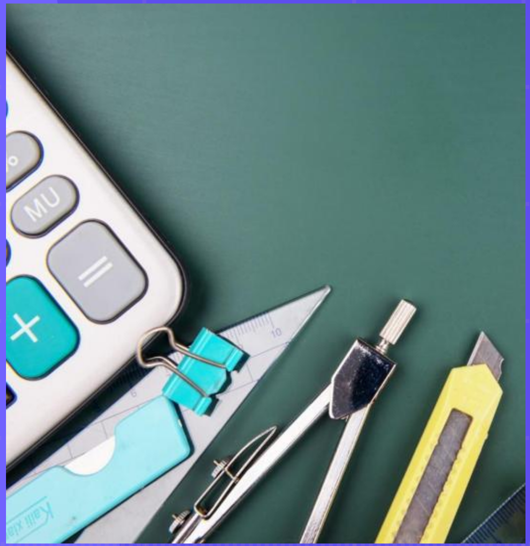


Three-dimensional vectors

To work with vectors in 3D we must use 3D space, therefore we use the x, y, z set of axes where all three axes are at right-angles to each other. The most common orientation for this space is:



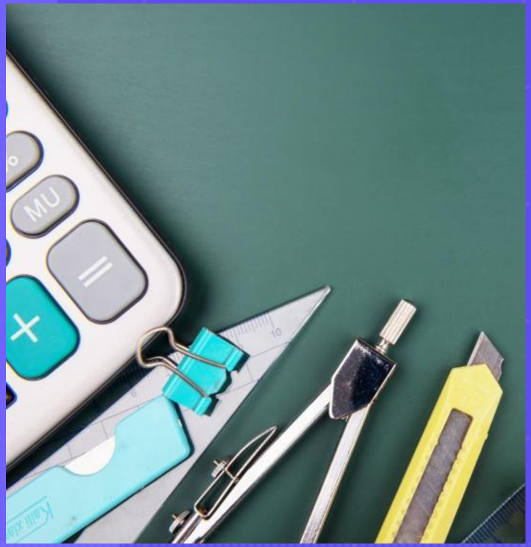
Using the $\boldsymbol{i}$ *and* $\boldsymbol{j}$ notation we use for vectors, we use $\boldsymbol{k}$ to represent a unit vector in the direction of z (mind blown!).



Three-dimensional vectors

When finding the magnitude of a vector in 3D space, not a lot has changed, we just add the third dimension into the mix:

$$|\boldsymbol{a}| = \sqrt{3^2 + 2^2 + (-5)^2}$$



Three-dimensional vectors

When finding scalar product, the formula doesn't change, but our interpretation of it does have to adjust a little.

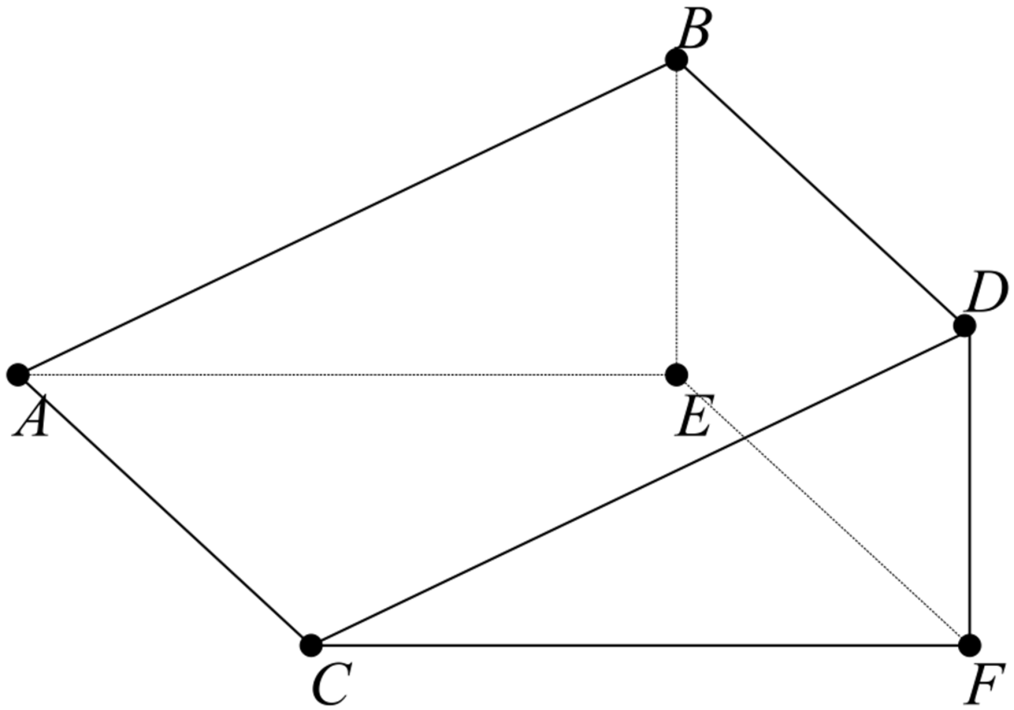
If $\boldsymbol{a} = < 3, 2, -5 >$ and $\boldsymbol{b} = < 1, 2, 3 >$

Then $\boldsymbol{a} \,.\, \boldsymbol{b} = (3)(1) + (2)(2) + (-5)(3) = -8$

We could then use $\boldsymbol{a} \,.\, \boldsymbol{b} = |\boldsymbol{a}| \times |\boldsymbol{b}| \times \cos\theta$

to find the angle between those vectors,

However, they may never intersect? Skew lines



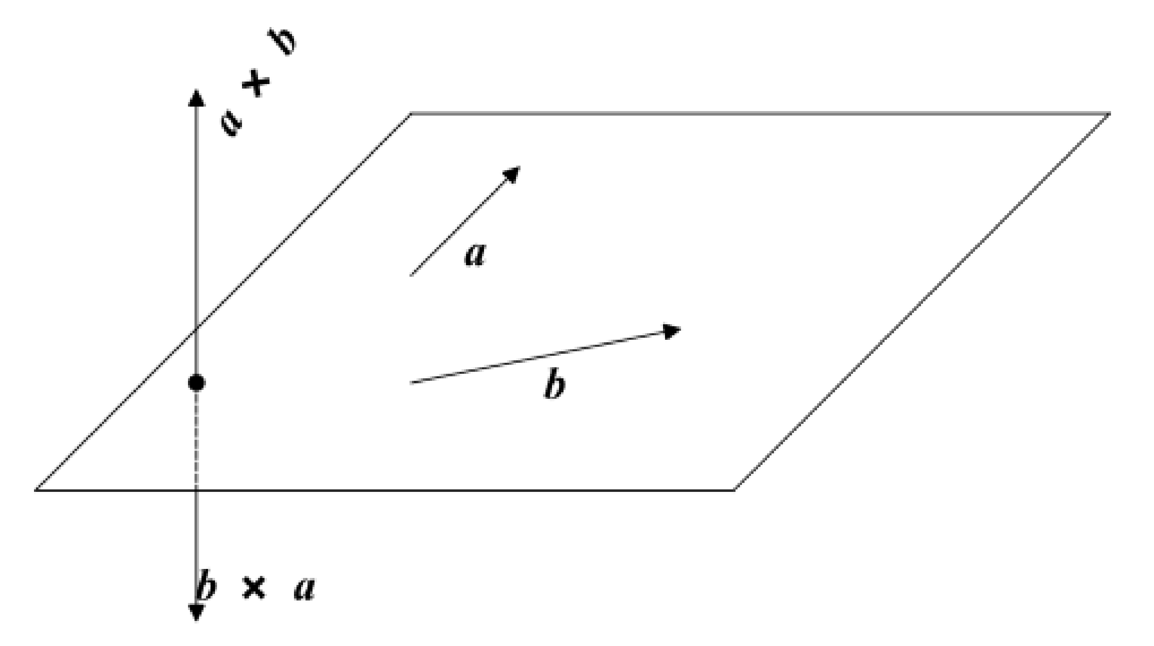
Angry product!

The cross product, or vector product, is the product of two vectors in space which result in a vector answer.

The cross product is written as $\boldsymbol{a} \times \boldsymbol{b}$

The cross product is extremely handy as the resultant vector is a vector which is perpendicular to the plane containing vectors a and b

Cross product	$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

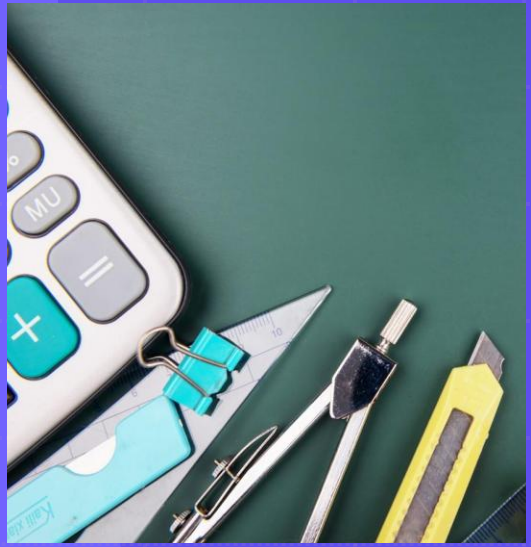


Angry product

Almost always any question requiring cross product are in the calculator section and hence you would use Classpad to find the normal to the plane from two vectors on the plane. This we could call the component form (ie the answer is given in i, j, k values)

Cross product also has a polar form

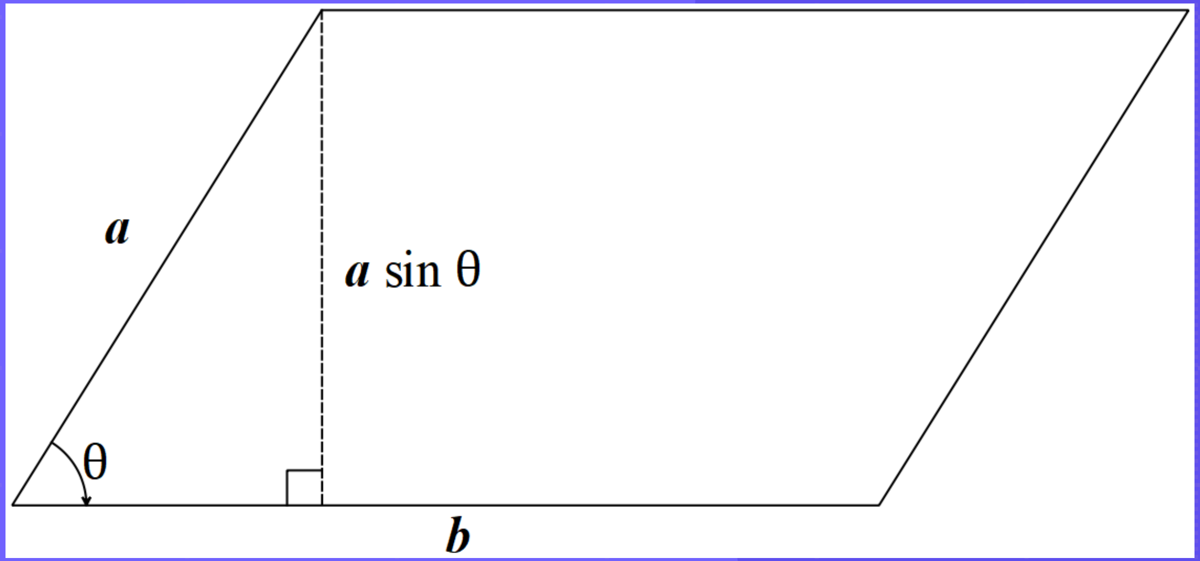
$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$**,** where θ is the angle between **a** and **b.**



The 'polar form' of the cross product produces the area of the parallelogram

Area by cross product

Consider the diagram given here.



$$
\begin{aligned}
Area &= base\ \text{'times'}\ height \\
&= |\boldsymbol{b}|\ \ \text{'times'}\ \ |\boldsymbol{a}|\sin\theta \\
&= |\boldsymbol{a}|\ |\boldsymbol{b}|\sin\theta
\end{aligned}
$$

This polar form creates a scalar answer (area). To create a vector (like we do when using component form) we need to multiply the formula above by a normal unit vector to create a vector that is normal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

e.g. to create a vector from the polar form:

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\ \ \hat{\mathbf{n}}$$

Vector equation of a line in 3D

Well not much changes!

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

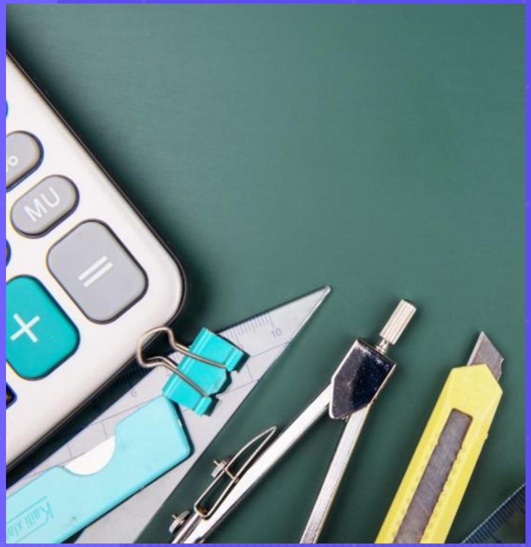
where $\boldsymbol{r}$ is some general point on the line.

$\boldsymbol{a}$ is the position of a point on the line.

$\boldsymbol{b}$ is a vector that the line is parallel to e.g. the direction of the line

Or we could use the scalar product format of a line $\boldsymbol{r} \,.\, \boldsymbol{n} = \boldsymbol{a} \,.\, \boldsymbol{n}$

Or in a simplified form: $\boldsymbol{r} \,.\, \boldsymbol{n} = c$ where $\boldsymbol{n}$ is a vector that is normal to line.



Parametric forms of a line

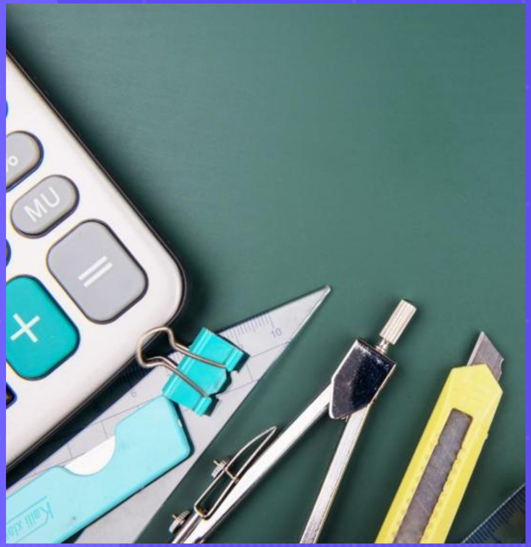
Consider the following line written in vector form:

$$\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

Writing this line in parametric form:

$$x = 3 + 2\lambda$$
$$y = 1 + \lambda$$
$$z = -1 + \lambda$$

Parametric equation of a line	$x = a_1 + \lambda u_1 \dots \dots (1)$ $y = a_2 + \lambda u_2 \dots \dots (2)$ $z = a_3 + \lambda u_3 \dots \dots (3)$



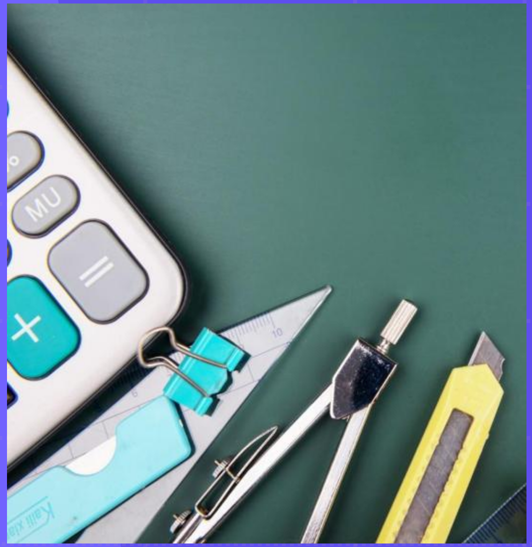
Planes in 3D

Defining a plane in 3D requires new thinking.

We can define a plane by:

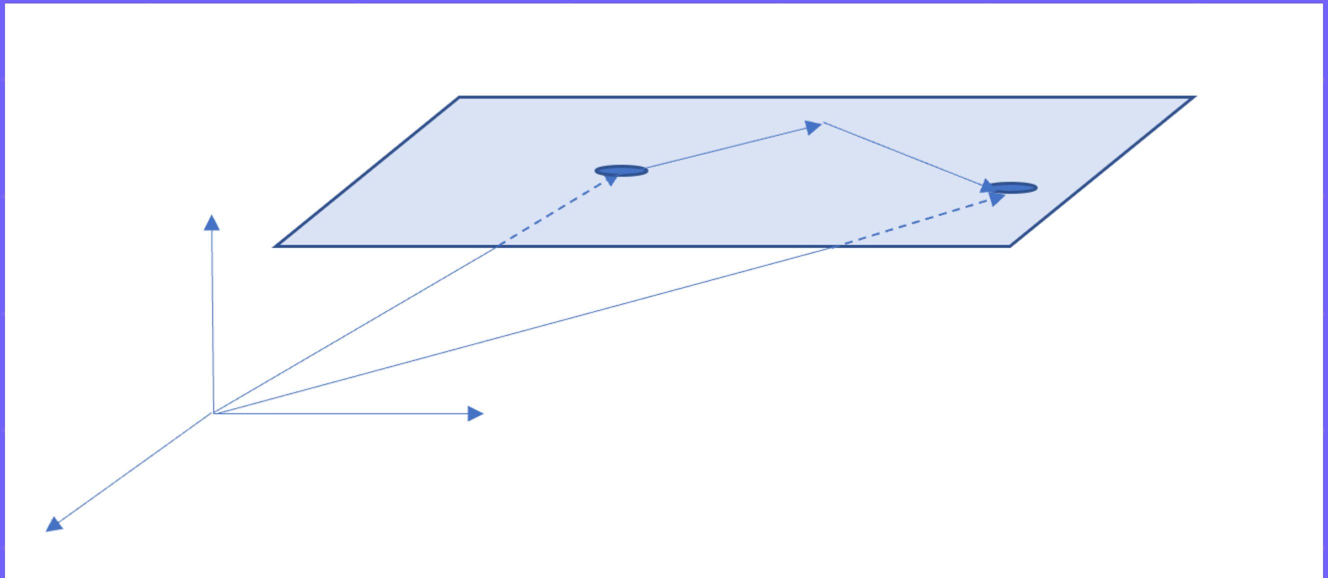
- One point and two nonparallel vectors on the plane
- Two points on the plane and the normal to the plane
- Three points on the plane

Regardless which we use, they all require the same basic information

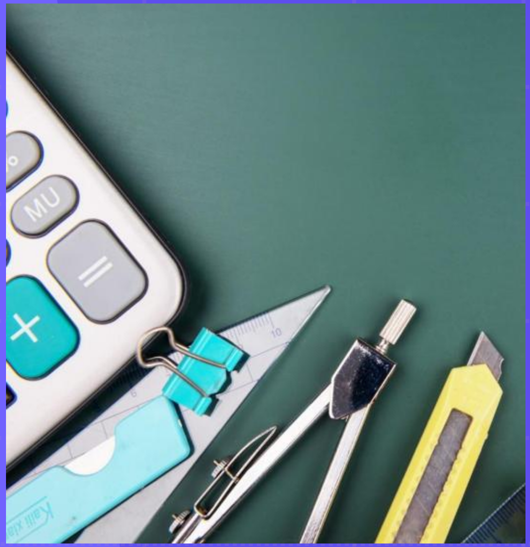


One point and two nonparallel vectors

We may also define a plane using a point that lies in the plane, A, and two vectors that are parallel to the plane, b and c



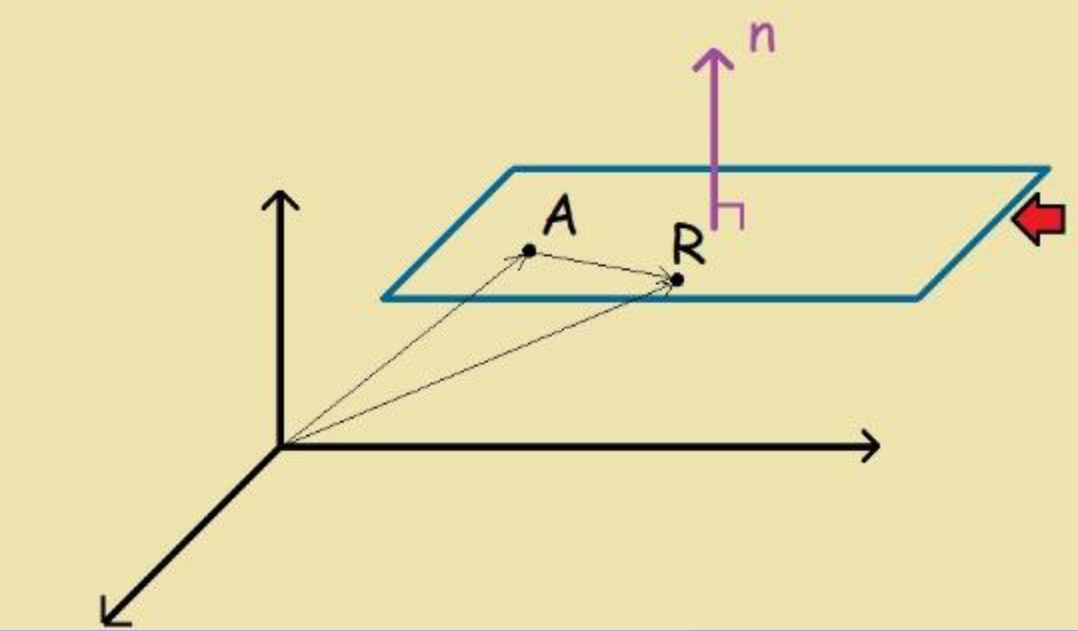
Hence $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ where $\boldsymbol{b}$ and $\boldsymbol{c}$ are not parallel.



Two points and the normal

If given two points on a plane and the normal, we can use this information to determine the equation of a plane. r-a will be a vector on the plane, and dotted to the normal must equal 0

$$\boldsymbol{n}\,.\,(\boldsymbol{r}-\boldsymbol{a})=0$$
$$\boldsymbol{n}\,.\,\boldsymbol{r}\;-\boldsymbol{n}\,.\,\boldsymbol{a}=0$$
$$\boldsymbol{n}\,.\,\boldsymbol{r}=\boldsymbol{n}\,.\,\boldsymbol{a}$$
$$\boldsymbol{r}.\boldsymbol{n}\;=c$$



n remains the normal, and c indicates the specific plane we are talking about. If we substitute a point for r, we can determine if it is on this plane

Cartesian form of a plane

Using our scalar product form of a plane

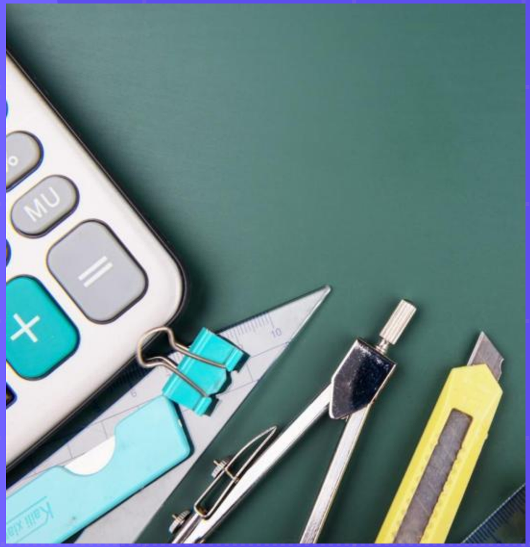
$$\boldsymbol{r}.\boldsymbol{n} = c$$

If we let $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $c = 2$, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2$$

$$x + 2y + 3z = 2$$

And hence we have the Cartesian equation of the plane



Formula sheet

Equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}_1 + \mu \mathbf{u}_2$ or $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

Cartesian equation of a plane	$ax + by + cz = d$

Vector equation of a sphere

The vector equation of a sphere with centre $\boldsymbol{d}$ and radius c has the equation:

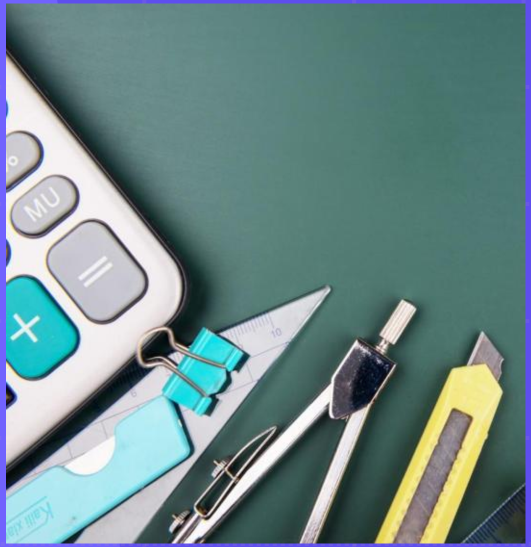
$$|\boldsymbol{r} - \boldsymbol{d}| = c$$

That is that the magnitude of any point subtracted away from the centre is equal to the radius. If a point does not conform to this equation, then it is not on the surface of the sphere.

We can then convert the vector equation to the Cartesian equation by writing $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$.

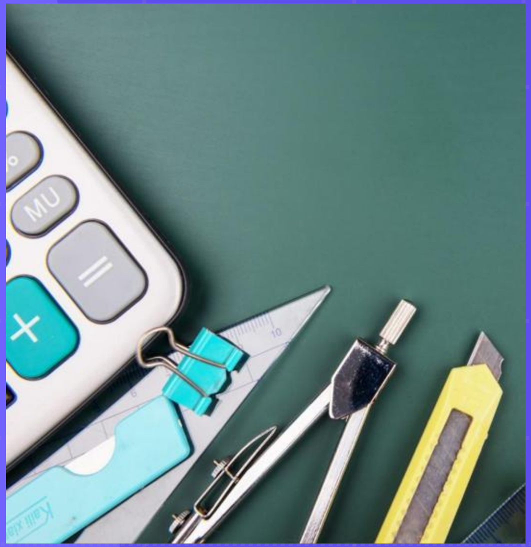
By doing this the Cartesian equation should be in the form:

$$(x-p)^2 + (y-q)^2 + (z-r)^2 = c^2$$



Vector equation of a sphere

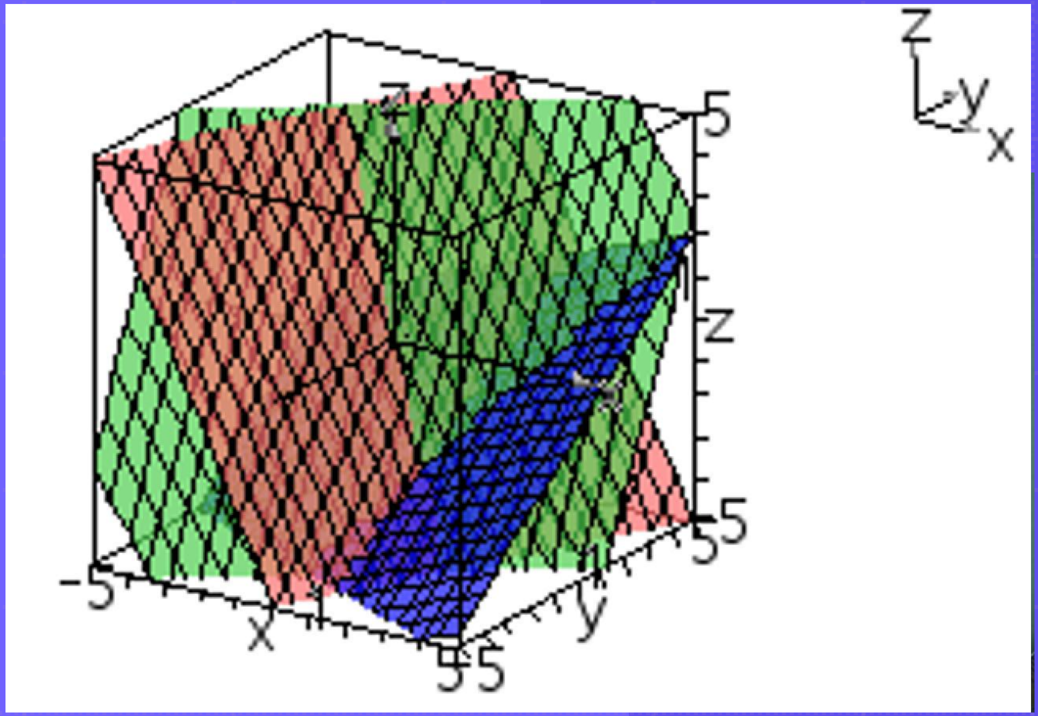
Equation of a sphere	$\lvert \mathbf{r} - \mathbf{d} \rvert = r$ or $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$



Systems of Equations – with a solution

In previous years of maths you would have used simultaneous equations many times to find the intersection point of two linear functions. Now that we are working in 3D space, how do we find the intersection point of three planes? Do the elimination and substitution method still work?

$$x + 2y - 3z = 9$$
$$2x - y + z = 0$$
$$-3x + 4y - 2z = 12$$



Of course yes, you now know this can be solved.

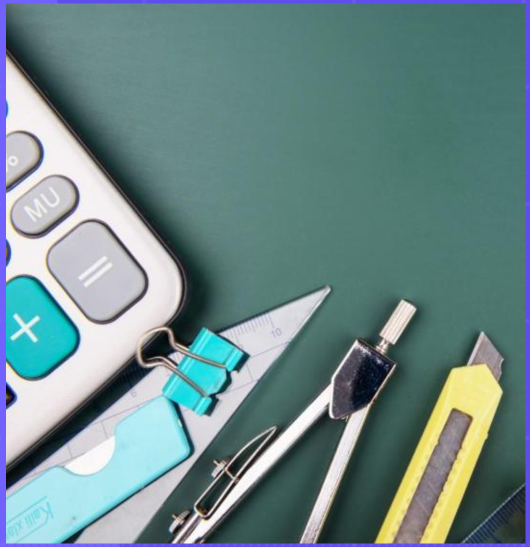
It is called 'system of equations'. Elimination and substitution will still work, but Gaussian elimination is quicker!

Systems of Equations - with a solution

These questions (well probably only one) is likely for the non-calc so, yes you do need to know how to do them! Watch out for the 'simple' case

Solve this system of equations.

$$x + y + 3z = 10$$
$$2x - y + z = 8$$
$$4x + y - z = 4$$

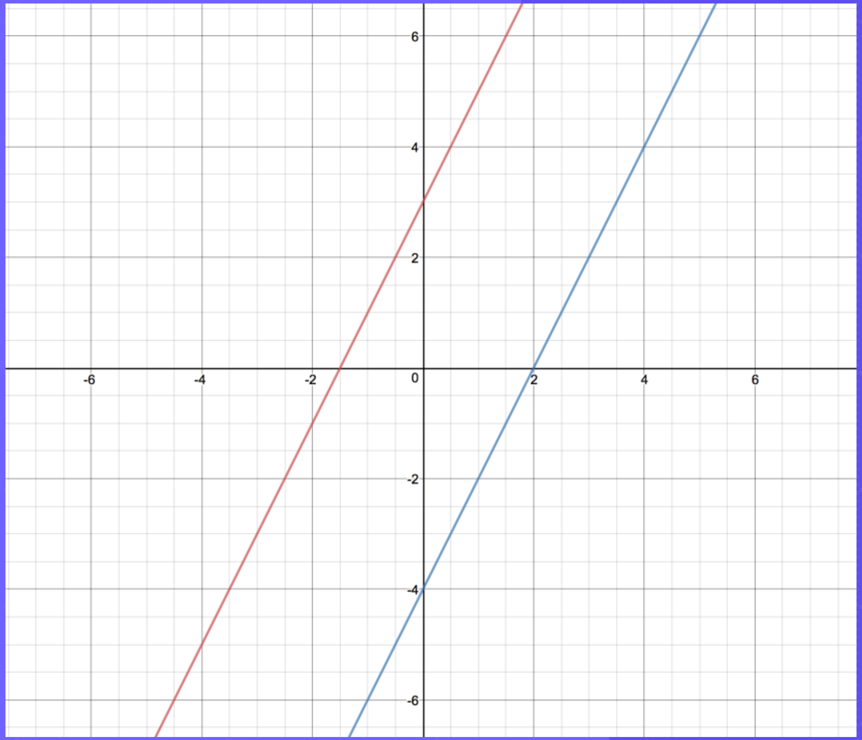


$$x = 2, y = -1, z = 3$$

Systems of Equations with no solution

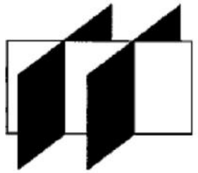
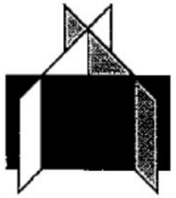
In a linear context no solution means the lines are parallel

$$y = 2x + 3$$
$$y = 2x - 4$$



In planes, no solution, would mean that at least two of the planes must be parallel or there are multiple points where two of the three planes intersect.

Systems of Equations with no solution

Three parallel planes.	One plane listed twice and one other parallel plane.	Two parallel planes and one other plane.	Intersecting pairs of planes form parallel lines.
	 Two equations defining the same plane.		
Example: $\begin{aligned} x-2y+3z &= 5 \\ -x+2y-3z &= 8 \\ 2x-4y+6z &= 7 \end{aligned}$	Example: $\begin{aligned} x-2y+3z &= 5 \\ x-2y+3z &= 8 \\ 2x-4y+6z &= 10 \end{aligned}$	Example: $\begin{aligned} x-2y+3z &= 5 \\ x+4y-2z &= 3 \\ 2x-4y+6z &= 11 \end{aligned}$	Example: $\begin{aligned} x+y-z &= 1 \\ x-2y-z &= -2 \\ -x+5y+z &= 7 \end{aligned}$
Augmented matrix can be reduced to:	Augmented matrix can be reduced to:	Augmented matrix can be reduced to:	Augmented matrix can be reduced to:
$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 6 & -5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$0x+0y+0z=13$!! $0x+0y+0z=-3$!!	$0x+0y+0z=3$!!	$0x+0y+0z=1$!!	$0x+0y+0z=1$!!
No solution.	No solution.	No solution.	No solution.

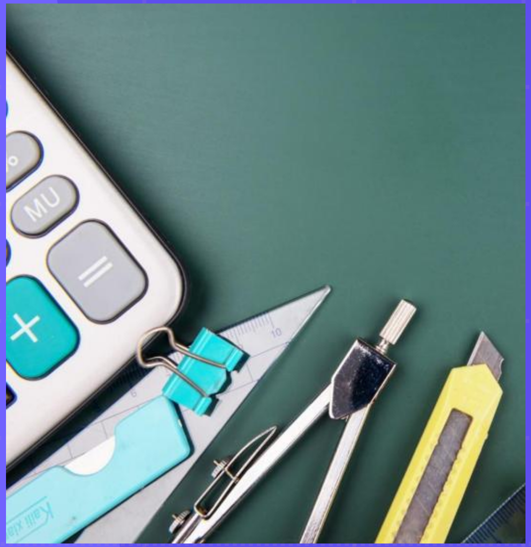
Systems of Equations with ∞ solutions

In a linear context ∞ solution means the lines are the same line

$$2x + y = 3$$
$$4x + 2y = 6$$

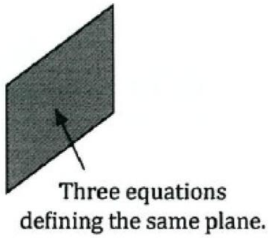
In this case they the second equation is multiplied by a linear factor of 2

In planes ∞ solutions would mean that there is at least a line (possible a plane) where all three planes exist.



Systems of Equations with ∞ solution

The same plane listed three times.



Example:

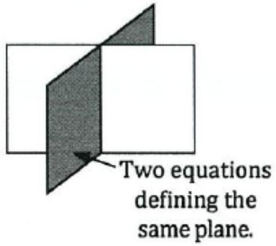
$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - 4y + 6z &= 10 \\ -x + 2y - 3z &= -5 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Sol^{ns} from 1st row.
Infinite solutions.

One plane listed twice and one other non-parallel plane.



Example:

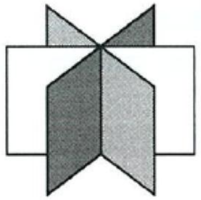
$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - 4y + 6z &= 10 \\ x + 3y + 3z &= 7 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Sol^{ns} from 1st & 2nd rows.
Infinite solutions.

Three planes meeting in a common line.



Example:

$$\begin{aligned} x + y - 2z &= -1 \\ x + 3y + z &= 0 \\ -2x - 4y + z &= 1 \end{aligned}$$

Augmented matrix can be reduced to:

$$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$0x + 0y + 0z = 0$ is uninformative.
Sol^{ns} from 1st & 2nd rows.
Infinite solutions.

Systems of Equations with multiple variables

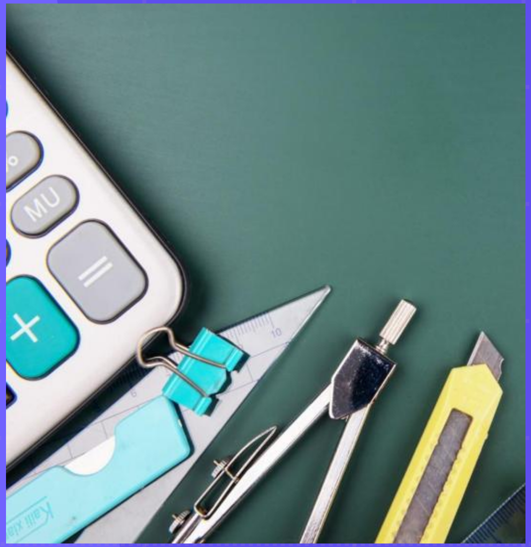
For example, determine the possible values p and q if the system of equations shown below has:

(a) no solution

(b) infinite solutions

(c) a unique solution

$$\begin{aligned} x - y + 2z &= 1 \\ 2x - 5y + 5z &= 9 \\ 3x + 3y + pz &= q \end{aligned}$$



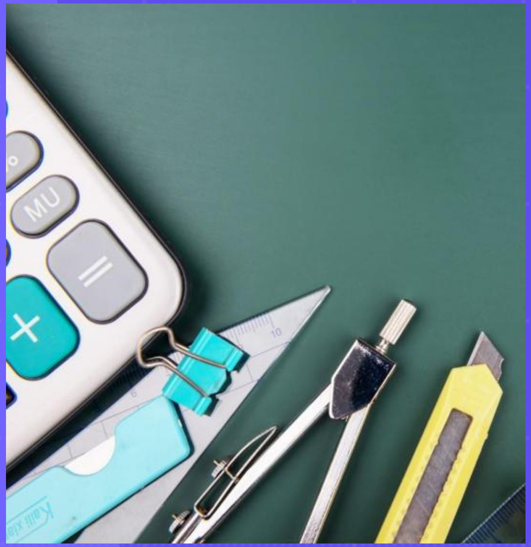
Systems of Equations with multiple variables

For these we do our usual Gaussian Elimination, carrying the variables through, and then once we have it to a Reduced Row Echelon Form (rref), we can then interpret the solution and determine what values p and q can take.

(a) no solution - $\mathrm{p} \neq 4, \mathrm{q}$ can take any value

(b) infinite solutions - $\mathrm{p} = 4, \mathrm{q} = -11$

(c) a unique solution - $\mathrm{p} = 4, \mathrm{q} \ \neq -11$

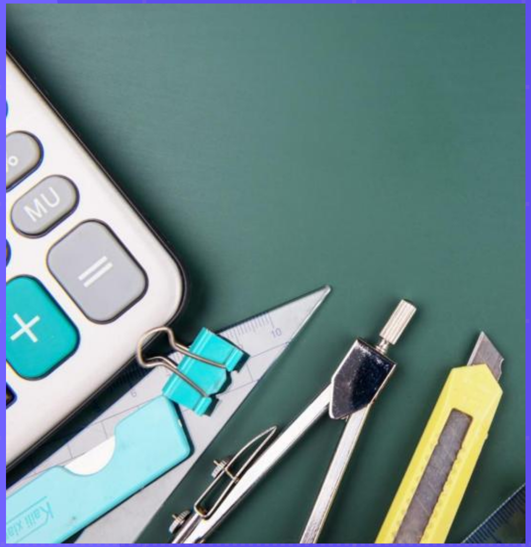


Vector calculus - differentiation

Consider a particle that moves along the line $\boldsymbol{r}(t) = < -2, 2 > \; + t < 1, 3 >$ where $\boldsymbol{r}$ is the particle's position (in metres) at some time t seconds ($t > 0$). Remember this is just an equation to describe motion, a starting point and a direction of travel. It is just a version of y = mx + c.

As $\boldsymbol{r}$ represents the particle's position, we can think of $\boldsymbol{r}$ as displacement. However, as we are now working in two directions, we can think of $\boldsymbol{r}$ as being displacement from the origin in <u>both the x and y directions</u>. Therefore, if we differentiate $\boldsymbol{r}$ with respect to time, the derivative function will be the <u>velocity function.</u>

Luckily, we do not have to convert vector functions back into Cartesian form to be able to differentiate them. To differentiate a vector function, we simply differentiate the $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ components separately <u>with respect to time</u>.



Vector calculus - differentiation

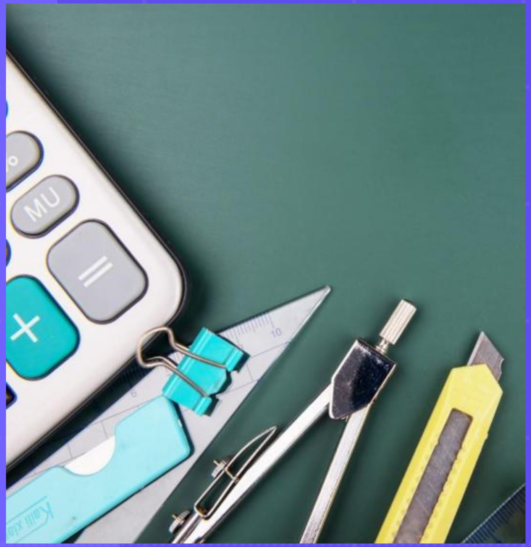
That is,

If $$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Then $$\frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}\right)$$

$$= \frac{df}{dt}\boldsymbol{i} + \frac{dg}{dt}\boldsymbol{j} + \frac{dh}{dt}\boldsymbol{k}$$

Which is the vector function for the velocity of the body. If we then differentiate again in this way (e.g. we determine the second derivative), we will determine the acceleration of the body in terms of time.



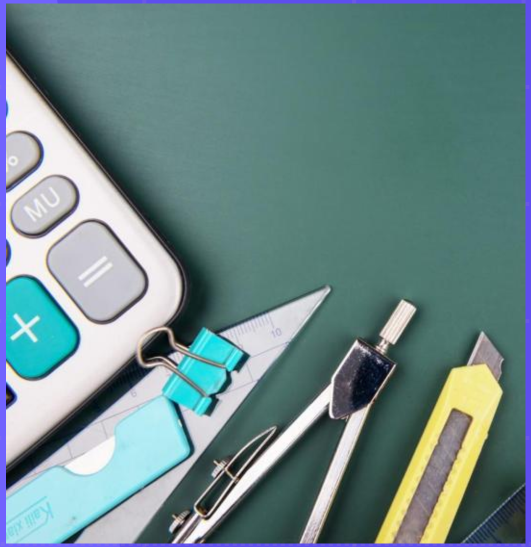
Vector calculus - differentiation

That is,

If $$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Then $$\frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}\right)$$

$$= \frac{df}{dt}\boldsymbol{i} + \frac{dg}{dt}\boldsymbol{j} + \frac{dh}{dt}\boldsymbol{k}$$



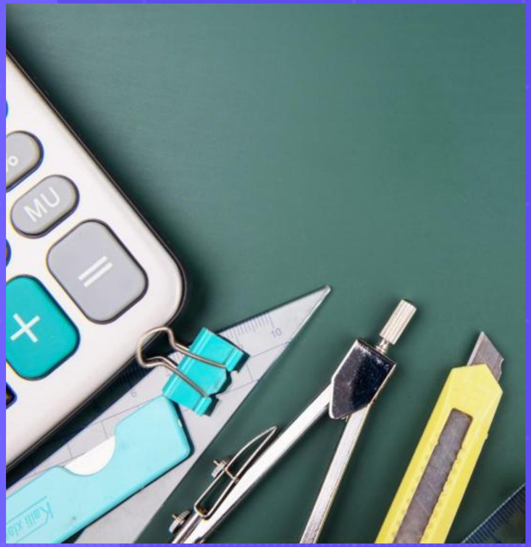
Which is the vector function for the velocity of the body. If we then differentiate again in this way (e.g. we determine the second derivative), we will determine the acceleration of the body in terms of time.

Vector calculus - integration

As differentiating vector functions is so simple (so far!) we should expect that integration of vector functions should be the same, but in reverse, right?

Well yes! To integrate vector functions, we simply integrate each component with respect to time. Remembering the relationship between position, velocity and acceleration.

To integrate acceleration, we would need a value for velocity (in component form), and to integrate velocity, we would need a value for displacement (in component form).



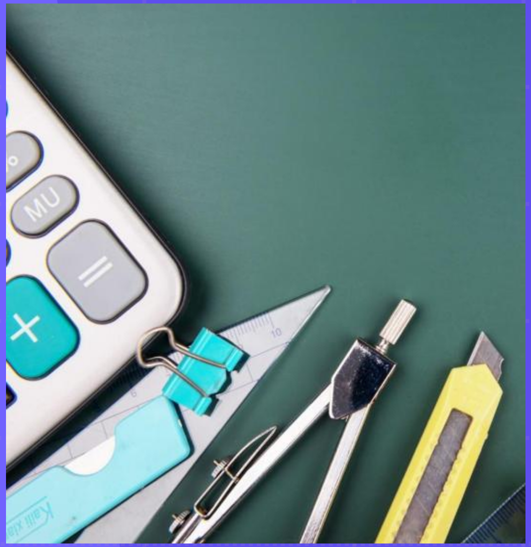
Vector calculus - Motion

The syllabus now says “apply vector calculus to motion in a plane, including projectile and circular motion”.

Some helpful formulas here are

During the time period between x and y:

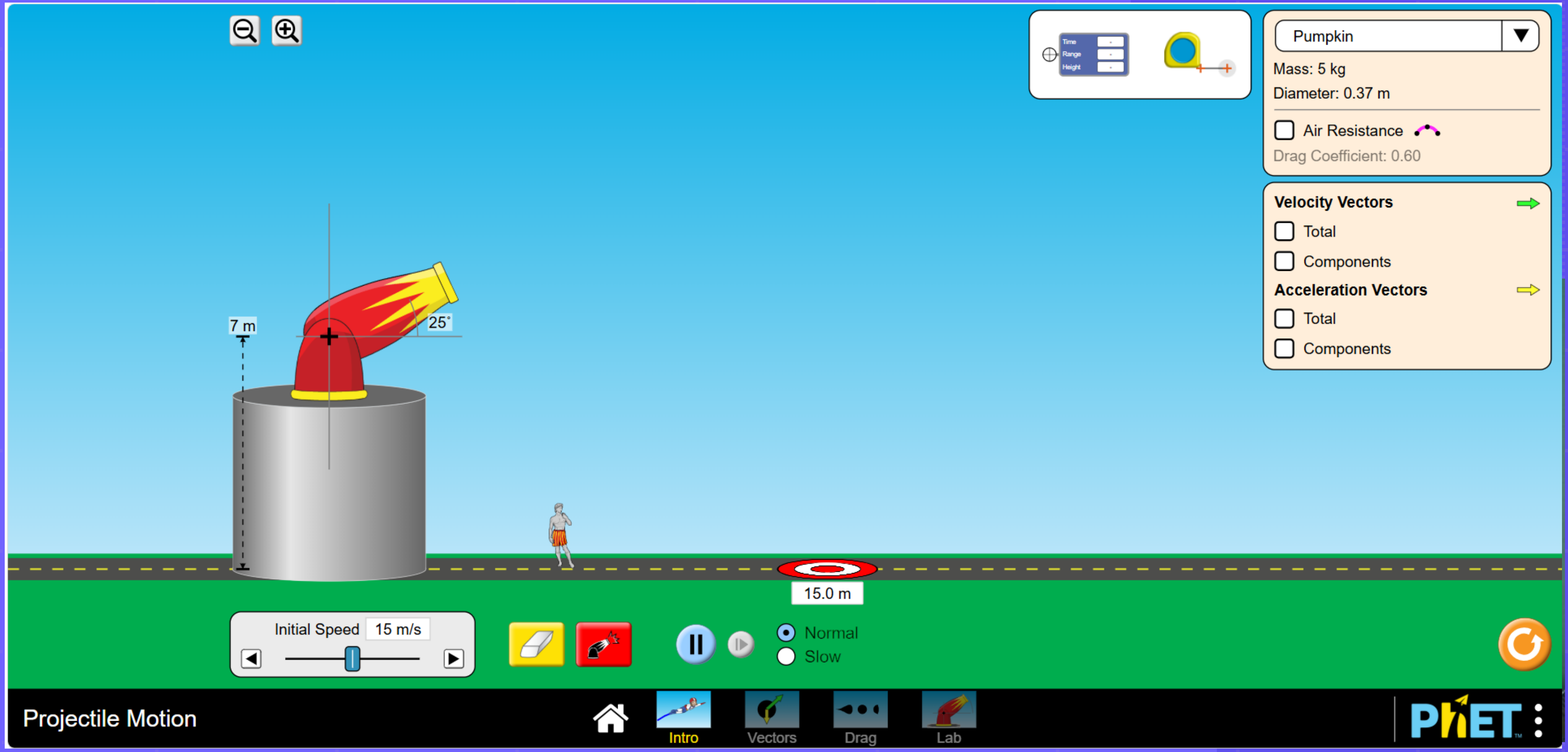
$\int_x^y \boldsymbol{v}\,(t)$	calculates the change in displacement.
$\lvert \int_x^y \boldsymbol{v}(t) \rvert$	calculates the magnitude of the change in displacement.
$\int_x^y \lvert \boldsymbol{v}\,(t) \rvert$	calculates the distance travelled along the path.

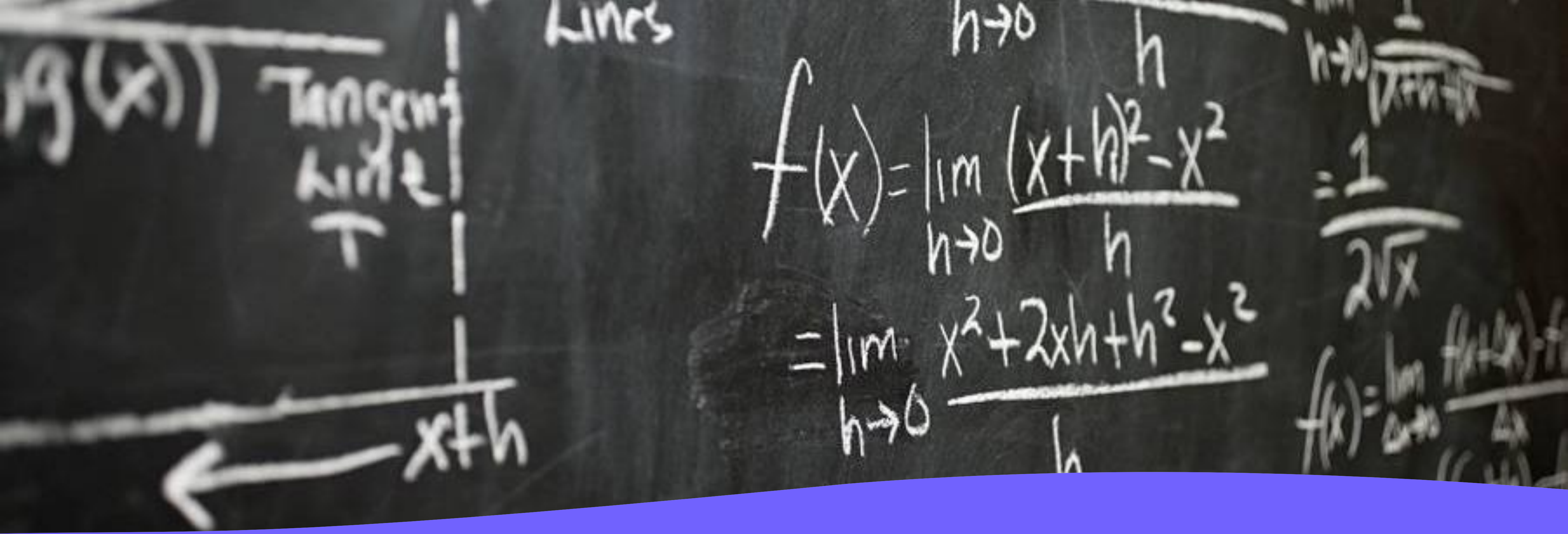


Vector calculus - Motion

This is a handy little one to have a play with for projectile motion

[Projectile Motion]





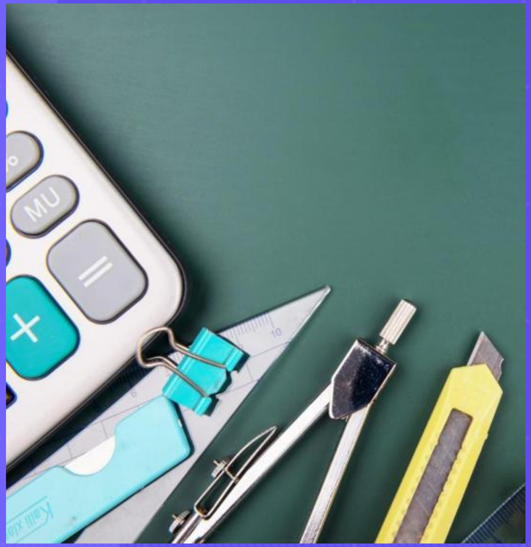
Questions?

WACE 2023 Question 9 (C)

Question 9 **(6 marks)**

The Cartesian equation of a sphere is given as $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$.

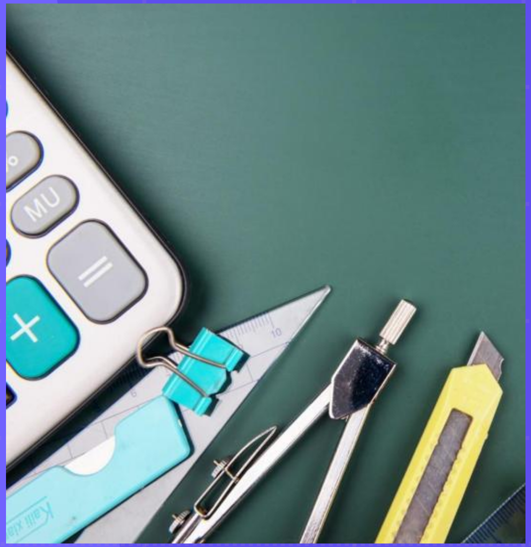
(a) Write the equation of the sphere in vector form. (3 marks)



WACE 2023 Question 9 (C)

A line has vector equation $\underset{\sim}{r} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

(b) Determine the point(s) of intersection between the line and the sphere. (3 marks)



WACE 2022 Question 5 (NC)

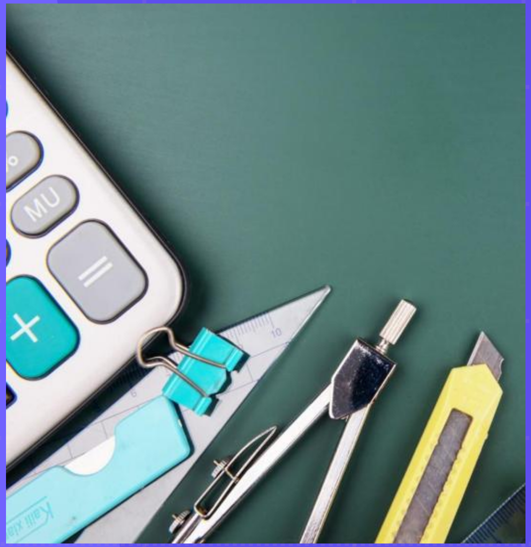
Question 5 **(6 marks)**

Consider the Cartesian equations for three planes:

$$\begin{aligned} 2x + 2y + z &= 9 \\ -2x + 2y - 5z &= -13 \\ y - z &= -1 \end{aligned}$$

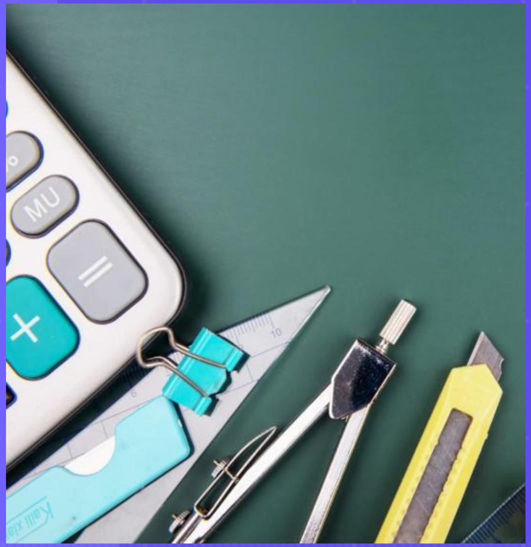
(a) Show that none of these planes is parallel to another. (2 marks)

(b) Solve the above equations simultaneously. (3 marks)



WACE 2022 Question 5 (NC)

(c) State the geometric interpretation of the solution obtained in part (b). (1 mark)



Unit 4 Topic 1

Integration and applications of integration

4.1.1 – 4.1.7

Syllabus points

Integration techniques

4.1.1 integrate using the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),\ \ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ and } 1 + \tan^2 x = \sec^2 x$$

4.1.2 use substitution u = g(x) to integrate expressions of the form *f(g(x))g'(x)*

4.1.3 establish and use the formula $\int \frac{1}{x}\, dx = \ln|x| + c$ for $x \neq 0$

4.1.4 use partial fractions where necessary for integration in simple cases

Syllabus points

Applications of integral calculus

4.1.5 calculate areas between curves defined by functions of the form $y=f(x)$ or $x=f(y)$

4.1.6 determine volumes of solids of revolution about either axis

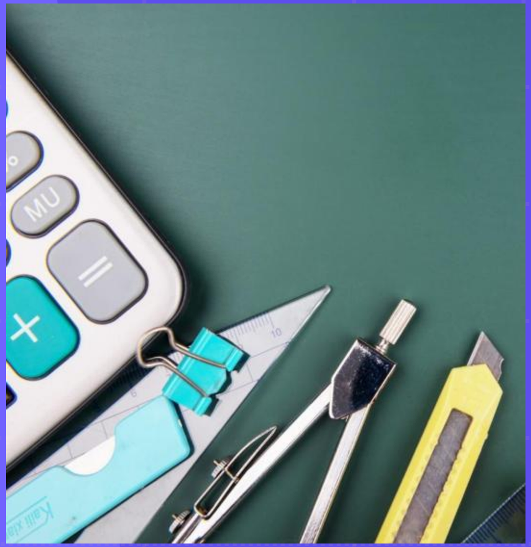
4.1.7 use technology to evaluate integrals numerically

Trig integrations

Some function can not be integrated as they are and require a trigonometric substitution.

These trig identities are mentioned in the syllabus

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $1 + \tan^2 x = \sec^2 x$



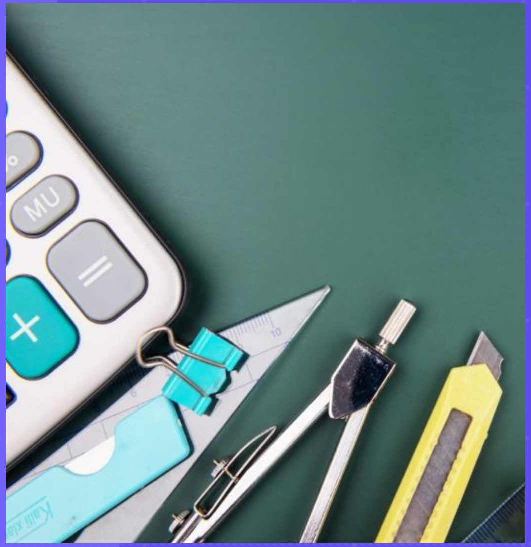
Trig integrations

Some function can not be integrated as they are and require a trigonometric substitution.

These trig identities are mentioned in the syllabus

$$\sin^2 x = \frac{1}{2}(1-\cos 2x),\ \ \cos^2 x = \frac{1}{2}(1+\cos 2x) \text{ and } 1+\tan^2 x = \sec^2 x$$

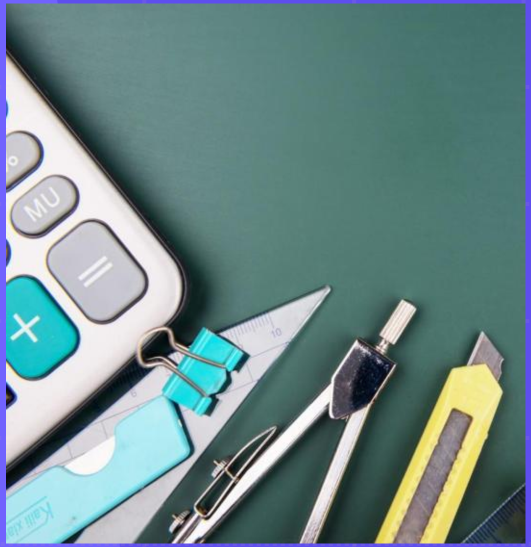
But in the formula sheet it gives us many more with which to be familiar.



Trig integrations

Year 12 Formula sheet

Identities	
$\cos^2 x + \sin^2 x = 1$	$1 + \tan^2 x = \sec^2 x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sin 2x = 2 \sin x \cos x$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
$\cos A \cos B = \dfrac{1}{2}(\cos(A - B) + \cos(A + B))$	$\sin A \cos B = \dfrac{1}{2}(\sin(A + B) + \sin(A - B))$
$\sin A \sin B = \dfrac{1}{2}(\cos(A - B) - \cos(A + B))$	$\cos A \sin B = \dfrac{1}{2}(\sin(A + B) - \sin(A - B))$



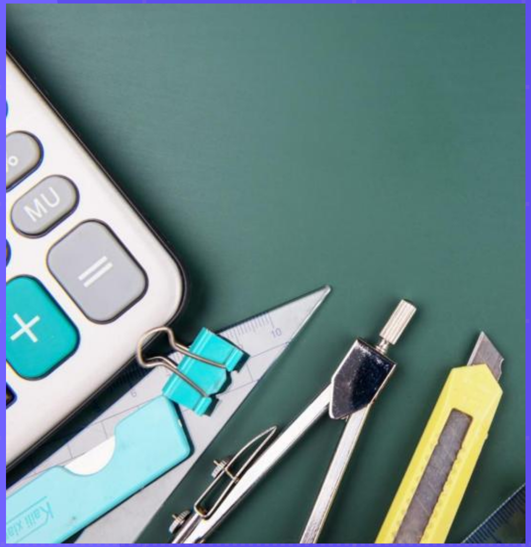
Trig integrations

And from Year 11 these may be needed ….

Basic trigonometric functions

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta \qquad \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$



Trig integrations

And….

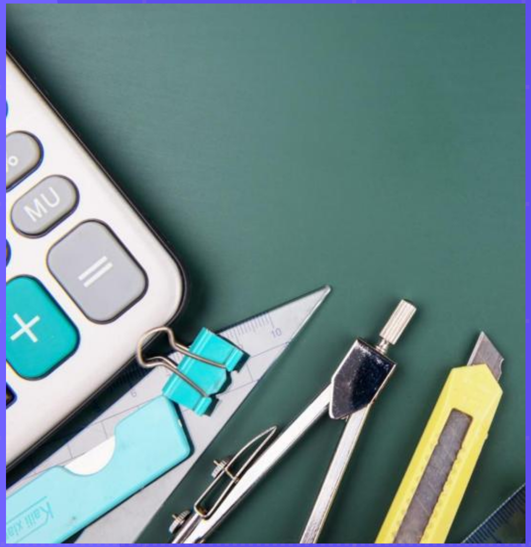
Compound angles

Angle sum and difference identites:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Double angle identities:

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$



Trig integrations

And….

Reciprocal trigonometric functions

$$\sec\theta = \frac{1}{\cos\theta},\ \cos\theta \neq 0 \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta},\ \sin\theta \neq 0 \qquad \cot\theta = \frac{1}{\tan\theta},\ \tan\theta \neq 0$$

Trigonometric identities

Pythagorean identities: $\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$

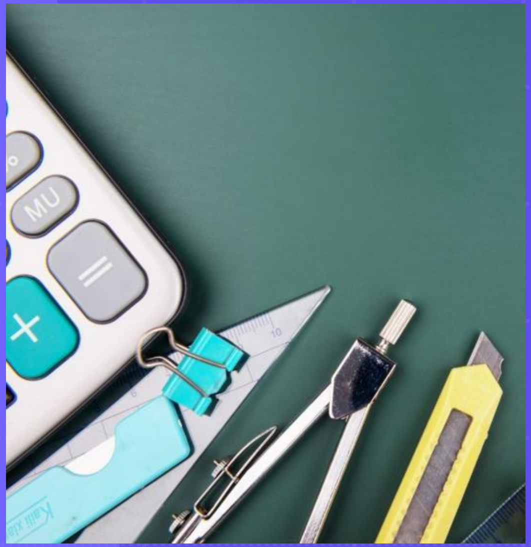
Product identities:

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$



Trig integrations

And….

Auxiliary angle formulae:

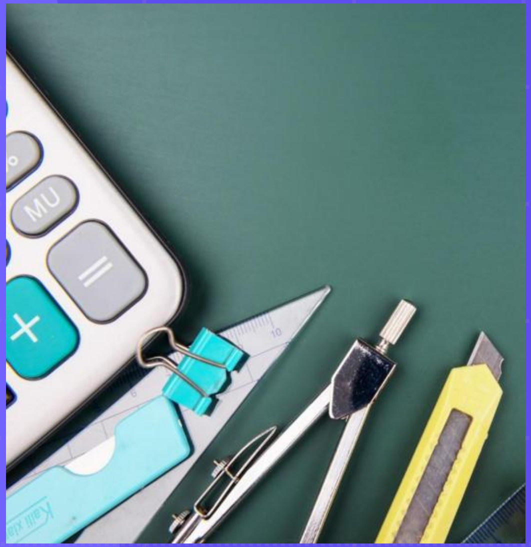
$$a \sin x \pm b \cos x = R \sin(x \pm \alpha) \text{ for } 0<\alpha<\frac{\pi}{2}, \text{ where } R^2 = a^2 + b^2, \tan\alpha = \frac{b}{a}$$

Triple angle identities:

$$\sin(3A) = 3\sin A - 4\sin^3 A$$

$$\cos(3A) = 4\cos^3 A - 3\cos A$$

$$\tan(3A) = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$



But these identities from Year 11 are unlikely to be needed (but just in case I have reminded you about them)

Trig integrations

Unfortunately, the syllabus is a bit vague on this point and so we must use past exams as our measure of what might be required and asked.

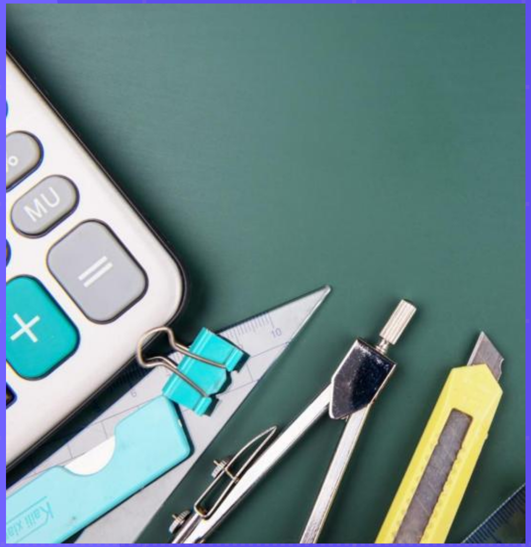
There are two that commonly come up and that is where we have an odd power, or an even power of a trigonometric expression.

Even powers of a trig function

Eg Determine $\int \cos^4 x \, dx$ we would use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Odd powers of a trig function

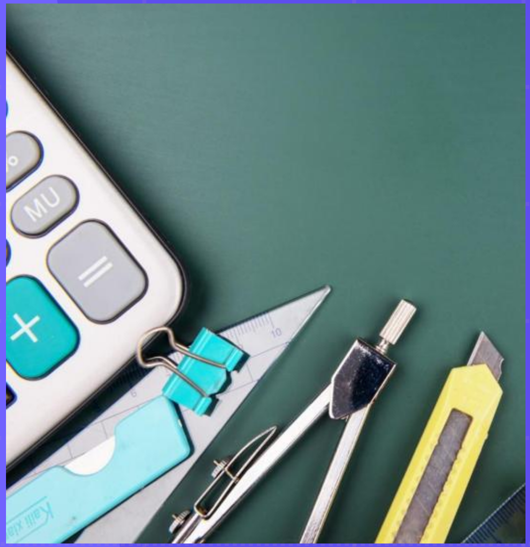
Eg Determine $\int \sin^5 x \, dx = \int (\sin x)(\sin^4 x) \, dx$ we would then use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$



Other Trig statements

Our formula sheet also has

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	Length of arc $= r\theta$
$a^2 = b^2 + c^2 - 2bc\cos A$	Area of segment $= \dfrac{1}{2} r^2 (\theta - \sin\theta)$
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$	Area of sector $= \dfrac{1}{2} r^2 \theta$



Integration by substitution

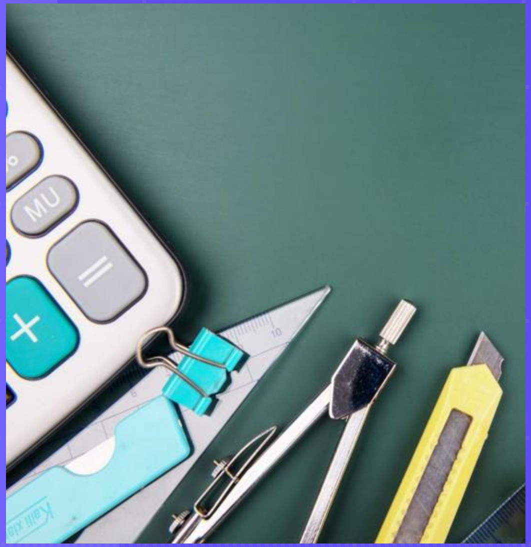
Consider the following question

Find $\int 18\mathrm{x}^2(x^3+6)^2\ dx$

From you Methods knowledge you will know that this integral is in chain rule form except for a scalar multiple e.g.

$$f'(x)[f(x)]^n$$

$$\int 18\mathrm{x}^2(x^3+6)^2\ dx = 2(x^3+6)^3 + c$$



Integration by substitution

But what about

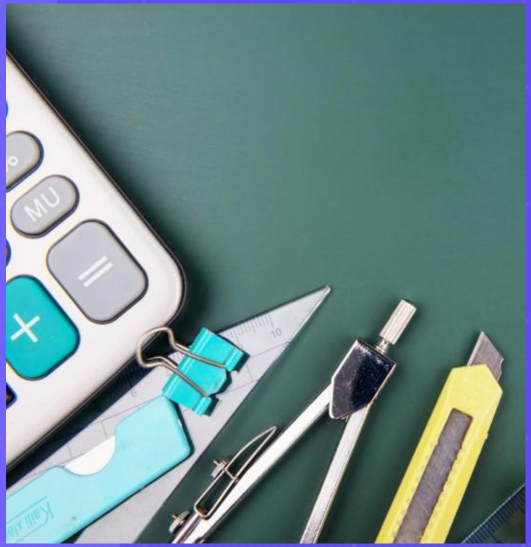
$$\int 12x\,(x+6)^2\ dx$$

This is no longer in the format of $f'(x)[f(x)]^n$ and we must use a subsitution to solve.

In this case,

let $\mathrm{u} = x + 6$ and hence $\mathrm{u} - 6 = x$ or $x = \mathrm{u} - 6$

And $\frac{\mathrm{du}}{\mathrm{dx}} = 1$ or $\mathrm{du} = \mathrm{d}x$



Integration by substitution

Now

$$\int 12x\,(x+6)^2\ dx$$

Substitute what you have

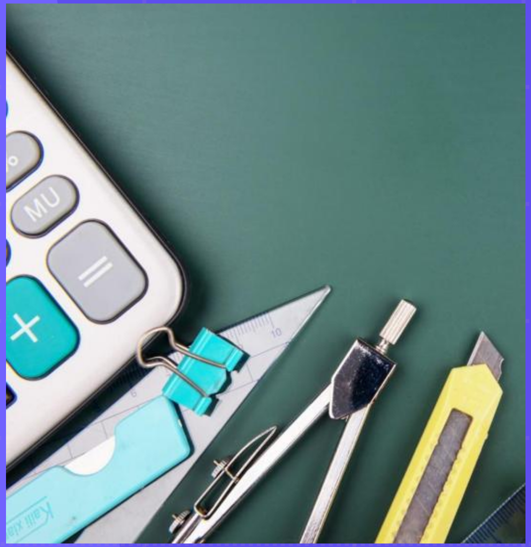
$$=\int 12(\mathrm{u}\ -6)\ (u)^2\ du.$$

$$=\int(12\mathrm{u}\ -72)\ (u)^2\ du$$

$$=\int 12\mathrm{u}^3\ -72\mathrm{u}^2\ du$$

$$=3u^4-\frac{72u^3}{3}+c$$

$$=3u^4-\frac{72u^3}{3}+c$$



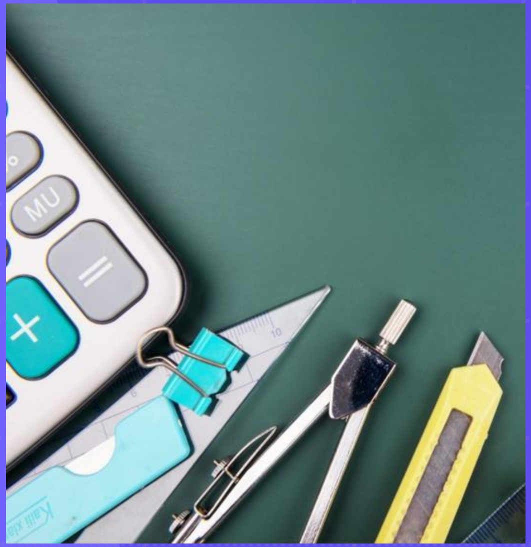
Integration of the natural logarithm

As you have already learnt how to integrate $\int \frac{1}{\mathrm{x}}$ in Methods, not a lot will be new here. In Specialist we now need to put absolute value signs around our integral.

That is $\int \frac{f\prime(x)}{f(x)} = \ln|f(x)| + c$, where we can express the numerator as the derivative of the denominator.

In the simplest case $\int \frac{1}{x} = \ln|x| + c$

In a more involved case $\int \frac{2x}{x^2+1}\ dx = \ln|x^2 + 1| + c$



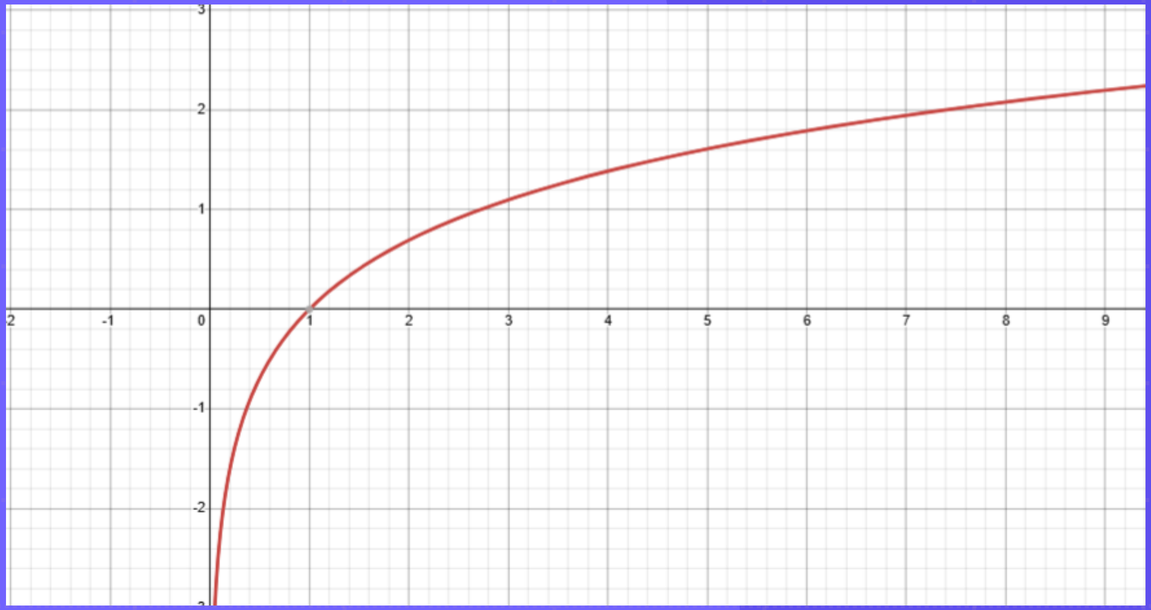
Integration of the natural logarithm

Why do we need the absolute value function for these integrals?   This is because the natural logarithm of a negative number is <u>undefined.</u>

Therefore, if $f(x)$ for some value of x formed a negative number (and we can't hope to test for all numbers possible across the domain), we need to put absolute value signs around our integral when introducing ln.

We must also exclude $x = 0$ as well

$$\int \frac{1}{x}\,dx = \ln|x| + c \text{ for } x \neq 0$$

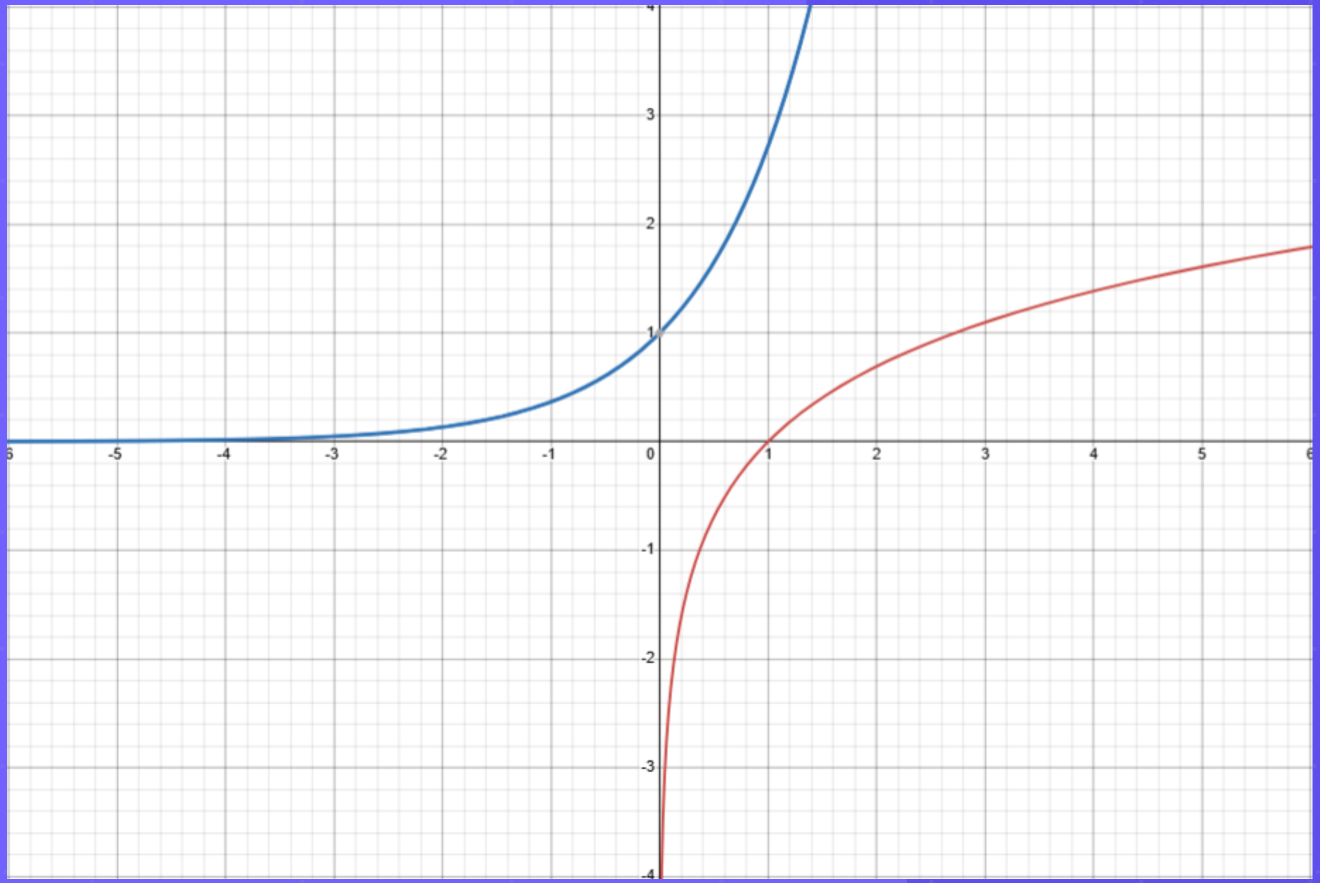


e^x v $\ln(x)$

Recap on the two functions which are the inverse of each other.

Inverse

- Swap x and y and rearrange), or
- Flips over the line y = x



$y = e^x$ (swap x and y)

$x = e^y$ (take ln of both sides)

$ln(x) = \ln(e^y)$ (take y out front)

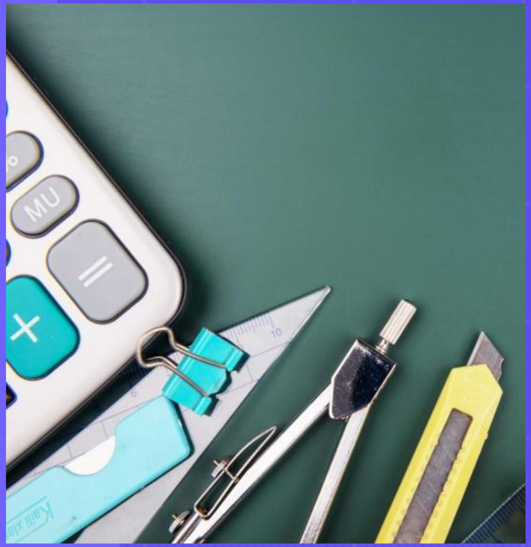
$ln(x) = y \ln(e)$ (ln(e) = log base e of e)

$ln(x) = y$ Or now, y = $ln(x)$, the inverse.

Integration and differentiation

Our formula sheet has a lot of facts; they don't really fit nicely into my presentation!

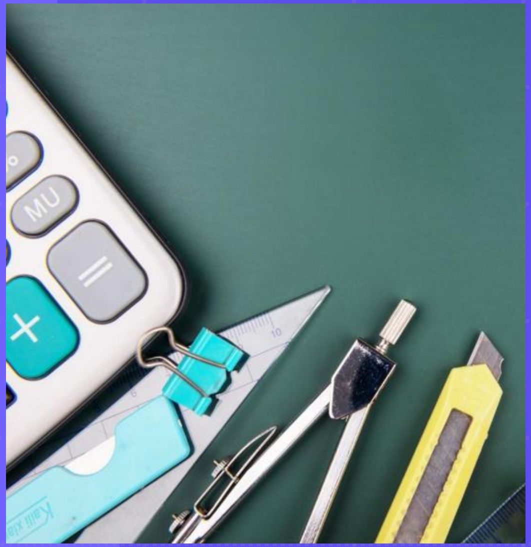
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
$\frac{d}{dx}e^{ax} = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$
$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$	$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$
$\frac{d}{dx}\sin f(x) = f'(x)\cos f(x)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\cos f(x) = -f'(x)\sin f(x)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\tan f(x) = f'(x)\sec^2 f(x) = \frac{f'(x)}{\cos^2 f(x)}$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$



Integration and differentiation

Our formula sheet has a lot of facts; they don't really fit nicely into my presentation!

Product rule	If $y = uv$ then $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$	or	If $y = f(x)\,g(x)$ then $y' = f'(x)\,g(x) + f(x)\,g'(x)$
Quotient rule	If $y = \frac{u}{v}$ then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	or	If $y = \frac{f(x)}{g(x)}$ then $y' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$
Chain rule	If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	or	If $y = f(g(x))$ then $y' = f'(g(x))\,g'(x)$
Fundamental theorem	$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$	and	$\int_a^b f'(x)\,dx = f(b) - f(a)$

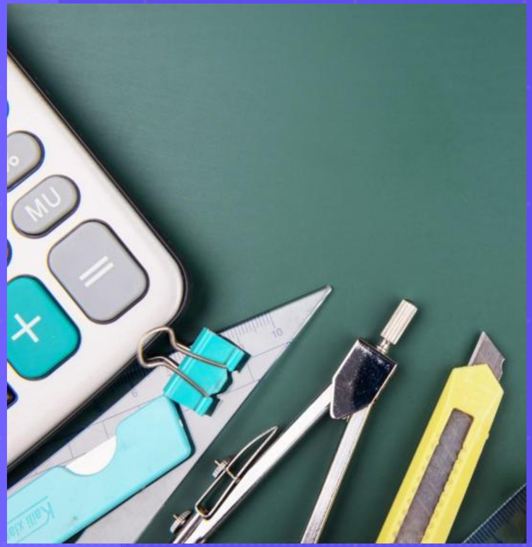


Partial fractions

Another way functions may not be able to be integrated is if they're in a format where the numerator and denominator are not derivates of each other, or can't be simplified to be in a suitable form.

A proper fraction → $\frac{a}{b}$ where $a < b$. Eg $\frac{3}{5}$ or $\frac{2x+5}{x(x+1)}$

An improper fraction → $\frac{a}{b}$ where b $< a$. Eg $\frac{7}{5}$ or $\frac{x^2+2}{x^2+6x+5}$ or $\frac{x^3+2}{x^2+6x+5}$



If we try to integrate some of these functions, it would be impossible to do, using our current knowledge of integration. We couldn't use substitution and we could not get them into the form $\int \frac{f\prime(x)}{f(x)}$ to use logarithmic integration

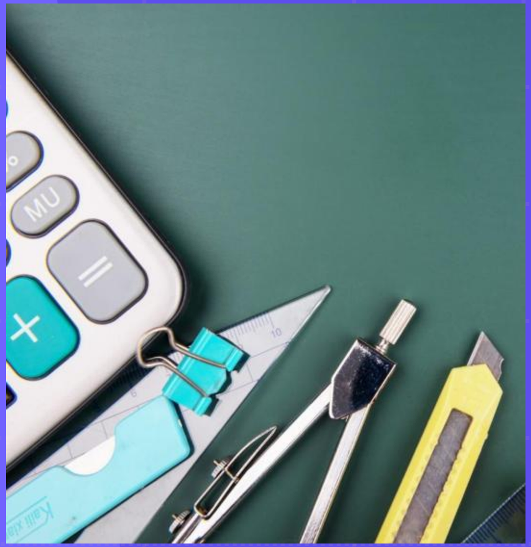
Partial fractions

So how do we deal with these?

A **proper fraction** can be decomposed depending upon their structure

- Unique Linear Factors in its denominator
- Repeated linear factors in its denominator
- Non-factorsiable quadratic factor in its denominator

An **improper fractions** may be decomposed into something simpler using algebraic long division or synthetic division



Partial fractions – Unique Linear Factors

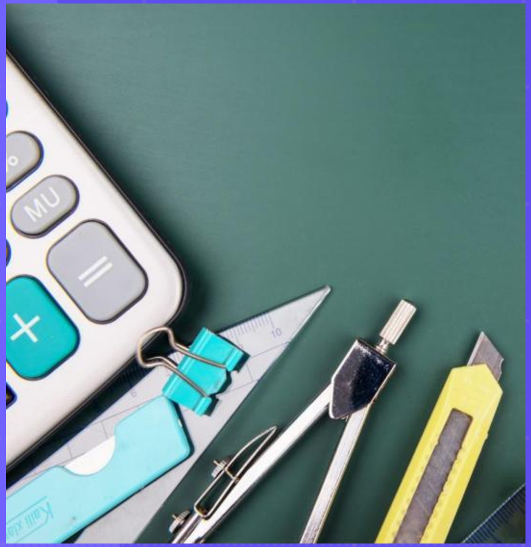
For proper fractions with unique linear factors in their denominator, for example

$\frac{2x+5}{x(x+1)}$ has unique linear factors, that is $(x+1)$ and x

Proper fractions with unique linear fractions in the denominator will decompose into partial fractions with each partial fraction having a one linear factor as its denominator.

$$\frac{2x+5}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

From here, it is a case of determining the values of A and B.



Partial fractions – Repeated linear factors

For a proper fraction that has repeated linear factors in its denominator, for example

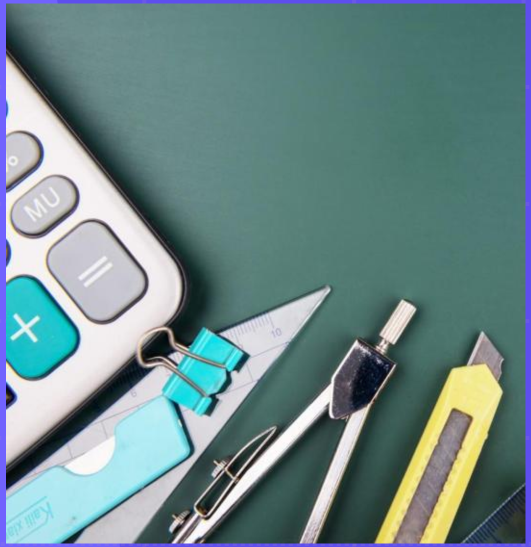
$$\frac{2x+5}{(x+2)(x+1)^2}$$

This type of proper fraction will decompose into partial fractions

$$\frac{2x+5}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

The general rule here is that you repeat the repeated linear factor, increasing the power by 1 each time.

$$\frac{x}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{R}{(ax+b)^n}$$

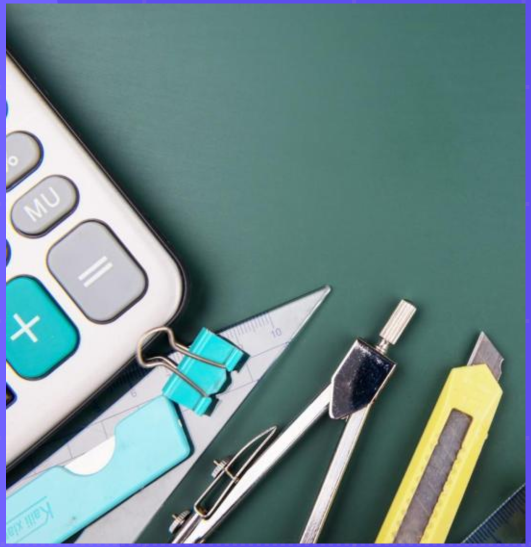


Partial fractions – Non-factorisable quadratic factor

The last style of proper fraction that we can encounter is one with a non-factorisable quadratic factor in its denominator.

$$\frac{x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Take note that this style of proper fraction has a linear expression over the non-factorisable quadratic factor.

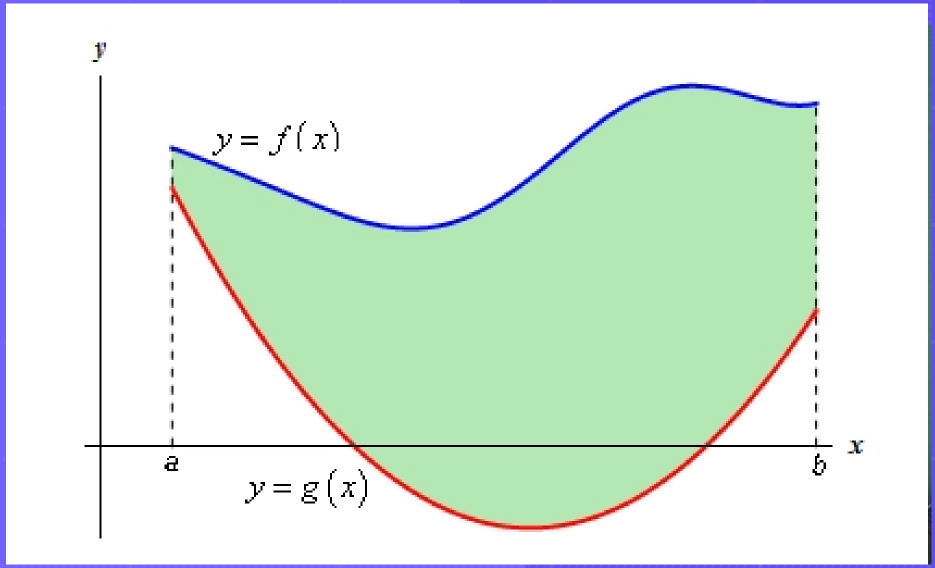


Area between curves

To find the area between two curves, you subtract the lower curve from the upper curve and integrate over the interval where that ordering holds. This ensures the area is non-negative.

$$A = \int_a^b f(x) - g(x)\ dx$$

It isn't always easy to know which function is above the other one for functions which cross over each other and so we can use



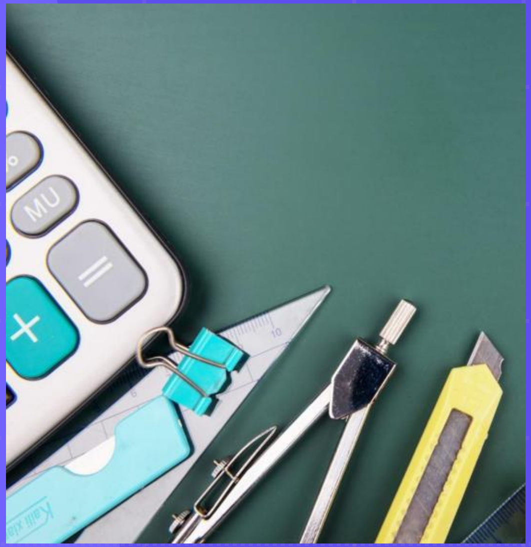
$$A = \int_a^b |f(x) - g(x)|\ dx$$

This will take the positive of any difference between the functions.

Volume of revolution

We have just seen definite integration finds the area under a curve between two bounds/limits. Volume of revolution takes this area and rotates it by a revolution around one of the axes to find a volume of revolution.

Let's see it…. Draft 3.1: Volume by Rotation with animation

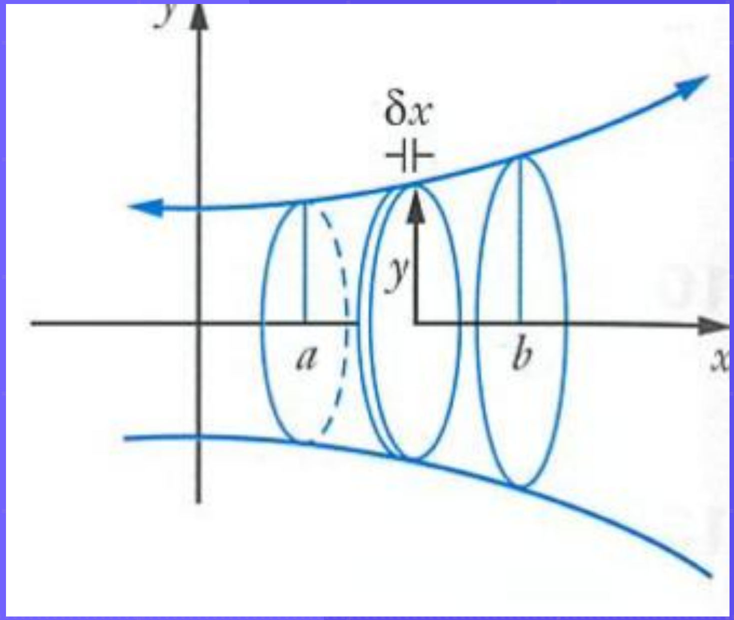


Revolution around the x-axis

Consider the diagram to the right.

If we want to determine the volume of the curved cylindrical type shape between a and b, we can consider breaking up this shape into infinitely small pieces with thickness δx. Hence, by using the volume of a cylinder formula

$$Volume\ of\ a\ cylinder = \pi r^2 h$$



substituting in y as the radius and δx height of the cylinder

$$Volume = \pi y^2\ \delta x$$

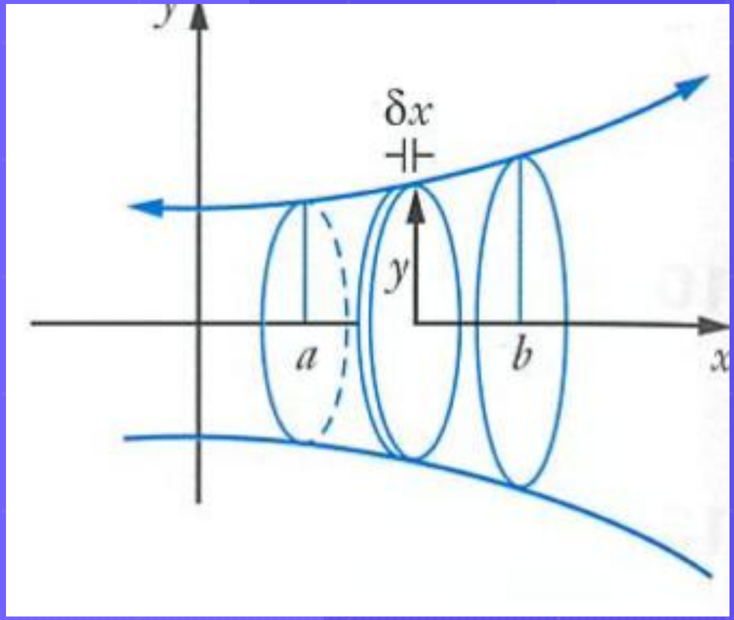
Revolution around the x-axis

Then considering the sum of the discs between a and b

$$Total\ Volume = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\ \delta x$$

Which we know this limit to be integration

$$Total\ Volume = \int_a^b \pi y^2\ dx$$



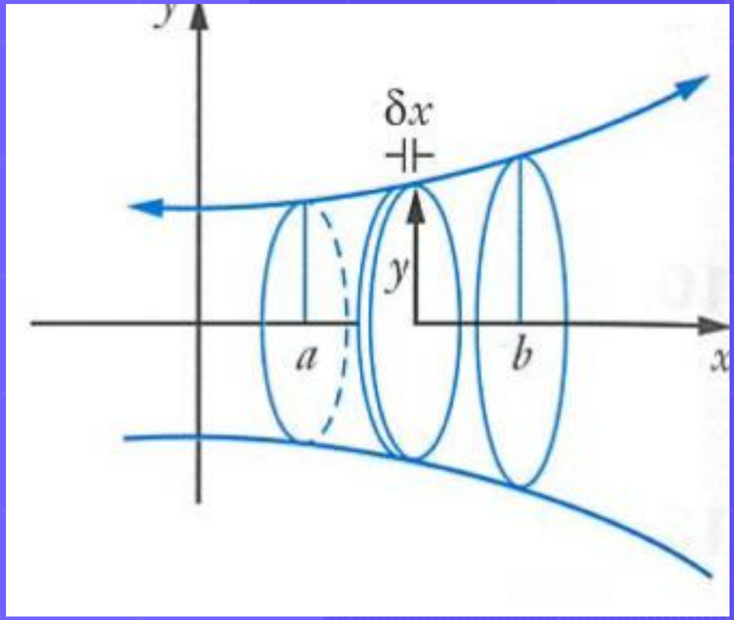
Revolution around the x-axis

Hence, to find the volume of revolution around the x-axis, we follow the formula

$$Total\ Volume = \int_a^b \pi y^2\, dx$$

The formula sheet has

About the x-axis	$V = \pi\int_a^b [f(x)]^2 dx$



Which seems different but $y = f(x)$, and it doesn't matter where we place π since π is a linear factor and is itself not a function.

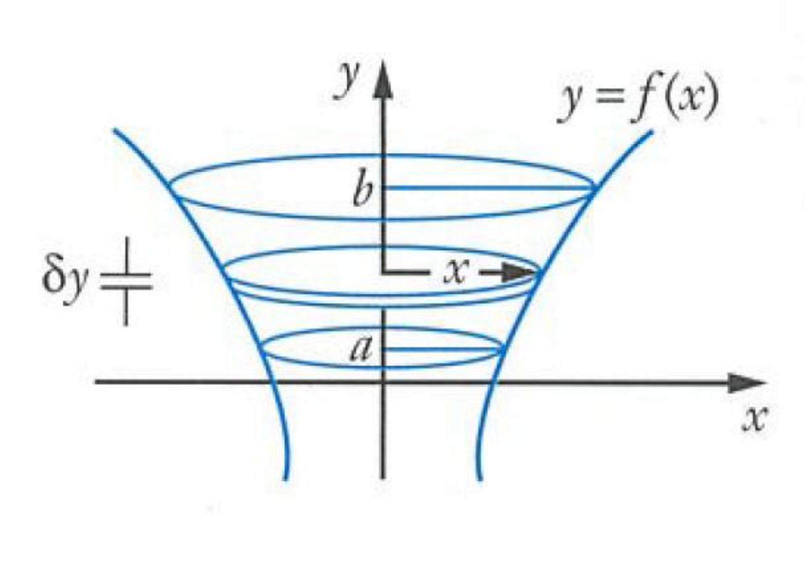
Revolution around the y-axis

Consider the diagram to the right.

If we want to determine the volume of the curved cylindrical type shape between a and b, but rotated around the y axis, in this case, the theory does not change much, but the function that we integrate does.

In this case, our radius becomes x, and our thickness of each disc becomes δy. Therefore, our volume of revolution is determined by:

$$Total\ Volume = \int_a^b \pi x^2\, dy$$



Revolution around the y-axis

And our formula sheet has

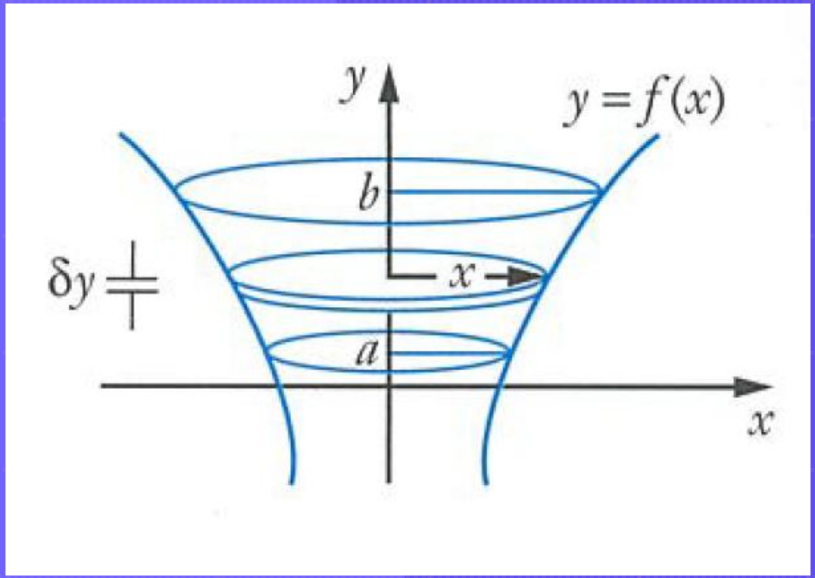
About the y-axis	$V = \pi \int_c^d [f(y)]^2 dy$

Again, this seems a mismatch for our formula of

$$Total\ Volume = \int_a^b \pi x^2\, dy$$

But again, $\mathrm{x} = f(y)$, and it doesn't matter where we place π

since π is a linear factor and is itself not a function



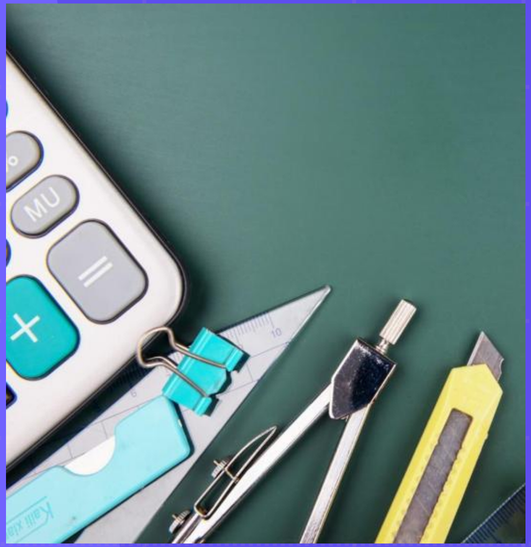
Last dot point in this section

4.1.7 use technology to evaluate integrals numerically

I wouldn't worry about this for the exam; it is extremely unlikely a question will be asking you to evaluate an integral numerically when you could actually just calculate the integral directly! exactly! simply! and accurately!

But for the matter of completeness of covering all the syllabus points…… two common approaches are

- Trapezoidal Rule
- Simpson's Rule (thank you, I'll take a bow now!)



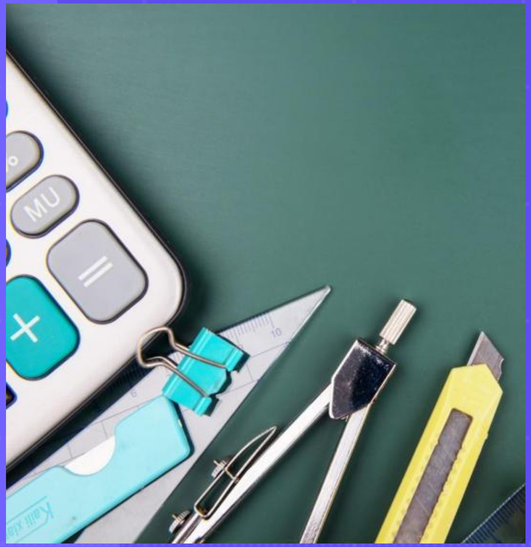
Trapezoidal rule

The Trapezoidal Rule is about breaking up the area under the curve into (you guessed it!), trapeziums!

If $y = f(x)$ was not easy to anti-differentiate we could find the area under the curve between a and b by breaking it up into trapezium strips of equal width. Therefore the total area:

$$\int_a^b f(x)dx \approx Area = h\left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \cdots + \frac{y_3 + y_4}{2}\right)$$

To understand that formula think of how the area of a trapezium is normally calculated.



Simpson’s rule 🥳

Simpson's rule looks at instead of creating trapezium strips, the curved end of the strip is replaced by a parabola that passes through the two edges of the strip and the midpoint of the strip. It creates an even number of strips.

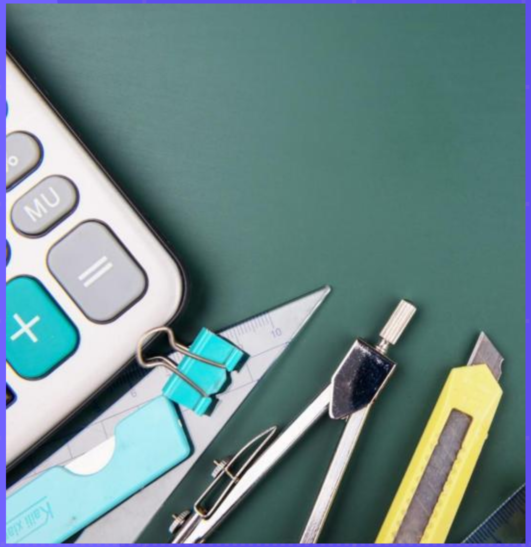
Let the number of strips/boxes be $2n$.

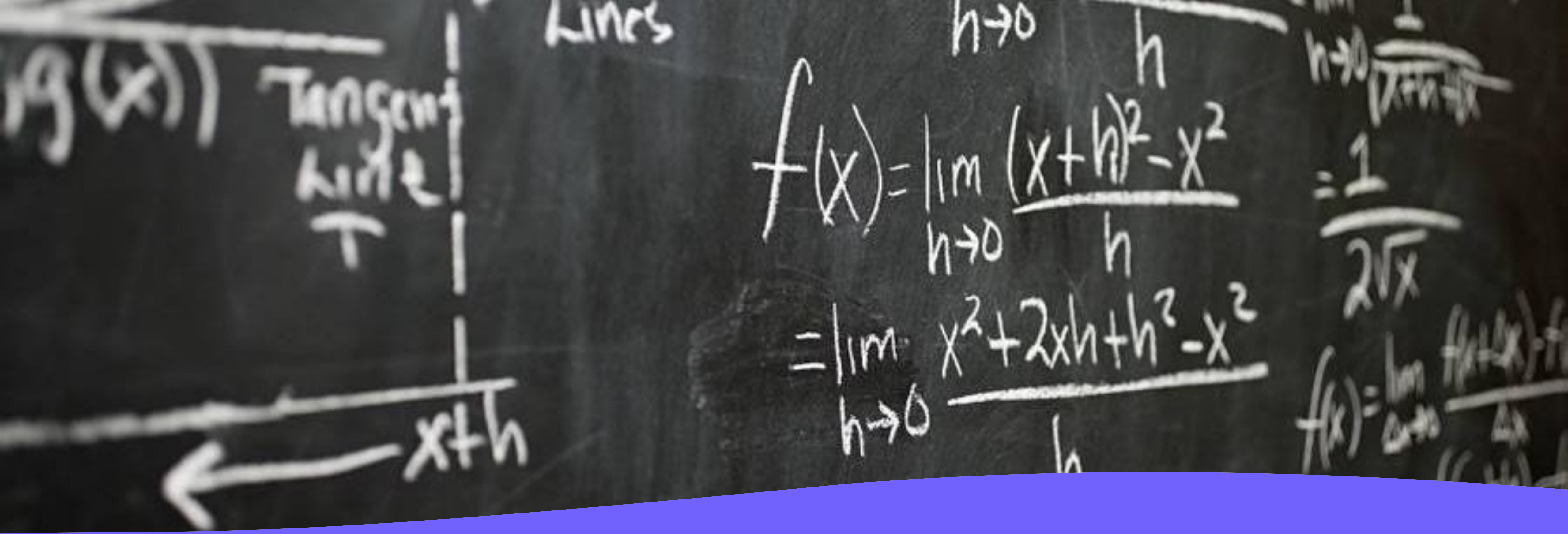
Width/height of each box/strip $w = \frac{b-a}{2n}$.

x coordinates where a new box/strip begins $x_{\mathrm{i}} = a + wi \quad for\ i = 0, 1, 2 \ldots 2n$

It can be shown (outside the scope of this course) that:

$$Area = \left(\frac{b-a}{6n}\right) \times [\, f(a) + f(b) + 4\sum_{i=0}^{n-1} f(x_{2i+1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) \,]$$





Questions?

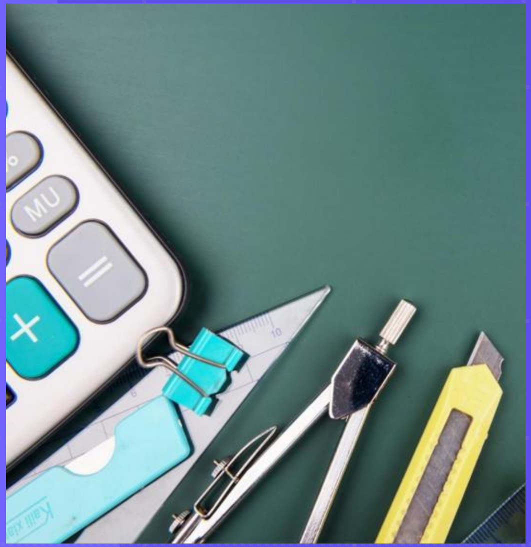
WACE 2022 Question 3 (NC)

Question 3 **(5 marks)**

By using one or more of the following identities:

$$\cos^2 x + \sin^2 x = 1$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2\sin x \cos x$$

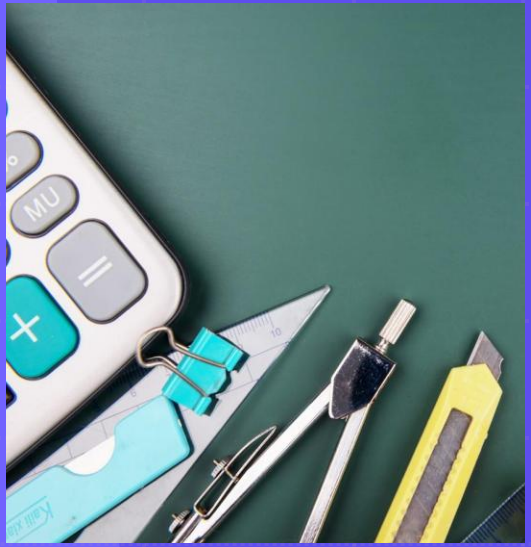
evaluate exactly $\displaystyle\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$.

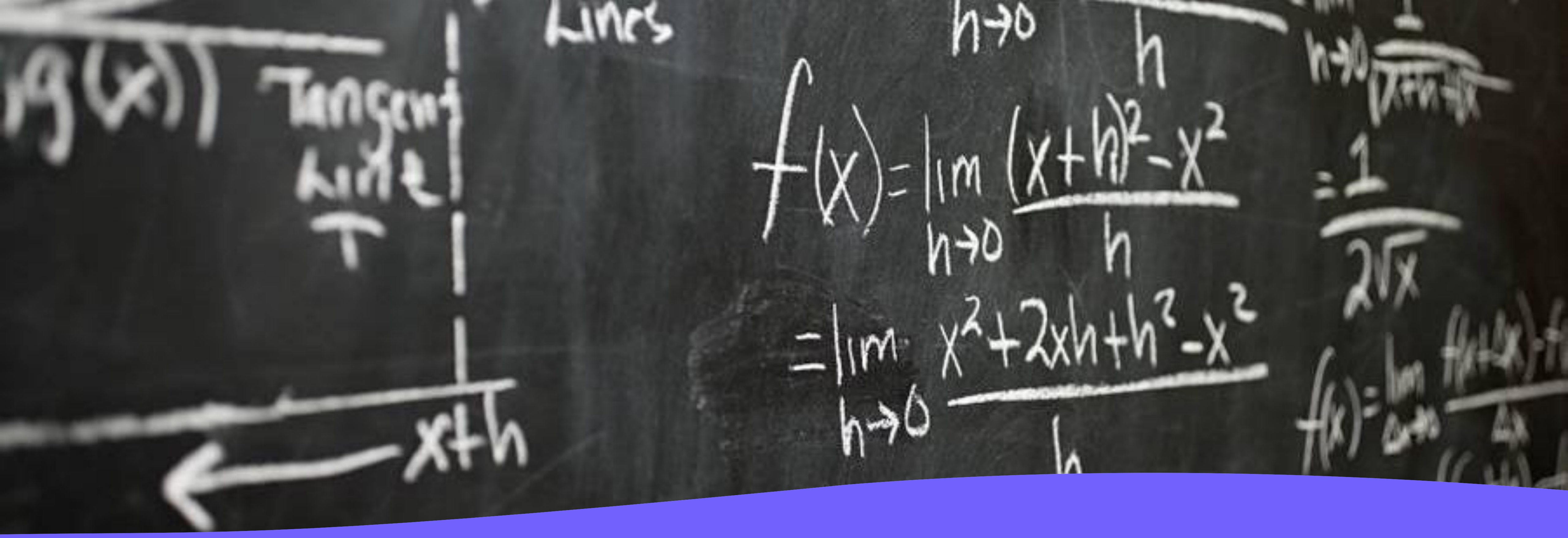


WACE 2023 Question 3 (NC)

Question 3 **(5 marks)**

Using the substitution $x = 119u + 1$, evaluate exactly $\displaystyle\int_1^{120} \left(2 + 4\left(\frac{x+118}{119}\right)^3\right) dx$.





Time to take a break!

10 minutes to stretch your legs, have a chat, use the restrooms!

Unit 4 Topic 2

Rates of change and differential equations

4.2.1 – 4.2.7

Syllabus points

Applications of differentiation

4.2.1 use implicit differentiation to determine the gradient of curves whose equations are given in implicit form

4.2.2. examine related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

4.2.3. apply the increments formula $\delta y \approx \frac{dy}{dx}\delta x$ to differential equations

4.2.4. solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$; differential equations of the form $\frac{dy}{dx} = g(y)$; and, in general, differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables

Syllabus points

4.2.5. examine slope (direction or gradient) fields of a first order differential equation

4.2.6. formulate differential equations, including the logistic equation that will arise in, for example, chemistry, biology and economics, in situations where rates are involved

Modelling motion

4.2.7. consider and solve problems involving motion in a straight line with both constant and non constant acceleration, including simple harmonic motion and the use of expressions, $\frac{dv}{dt}, v\frac{dv}{dx}$ and $\frac{d}{dx}(\frac{1}{2}v^2)$ for acceleration

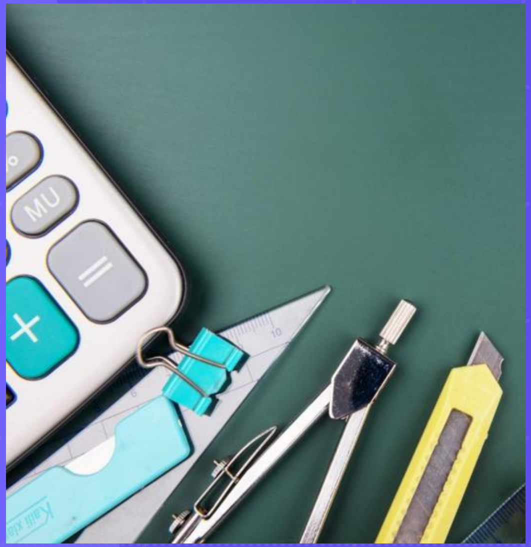
Implicit differentiation

Up until now (well Spec Semester Two) all functions that you have differentiated/integrated have been explicitly defined, for example, they have all been given with y as the subject which has been written explicitly in terms of x

For example, the function:

$$y = x^2 + 2x + 1$$

This function has y being explicitly defined in terms of x.

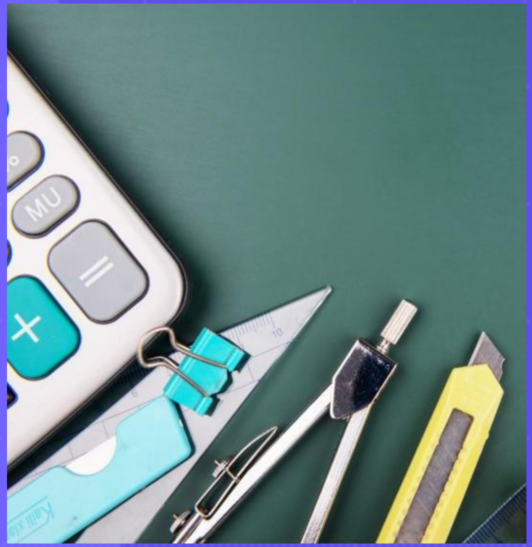


Implicit differentiation

Therefore, an implicitly defined function is where y (or some other variable) is not defined just in terms of x (or some other variable). For example:

$$3x + 2y = 6y + 1$$

This is an implicitly defined function. We cannot differentiate this function the same as usual as we cannot differentiate one side with respect to x only.



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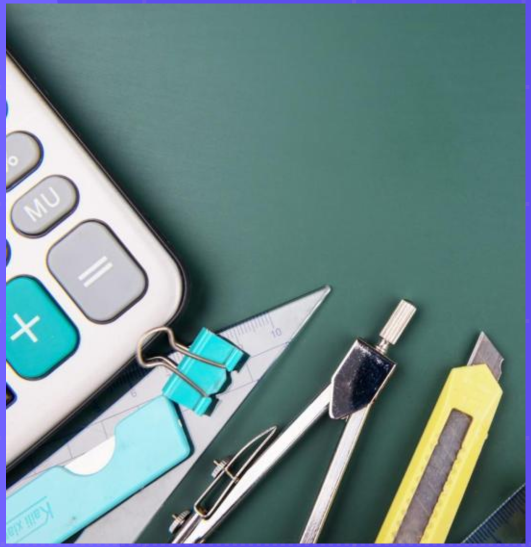
So, can't we just rearrange implicit functions into explicit functions? Yes, sometimes, no, maybe…..



Implicit differentiation

Let's work with this function

$$3x + 2y = 6y + 1$$

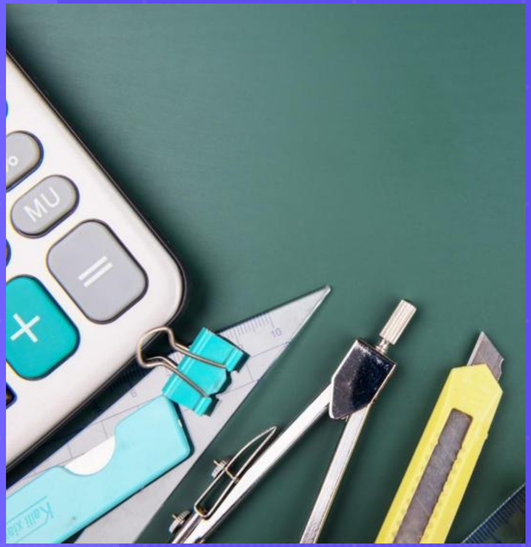
Let's take $\frac{\mathrm{d}}{\mathrm{dx}}$ of both sides (note I'm not defining my numerator variable)

$$\frac{\mathrm{d}}{\mathrm{dx}}(3x + 2y) = \frac{\mathrm{d}}{\mathrm{dx}}(6y + 1)$$

Expand $$\frac{\mathrm{d}}{\mathrm{dx}}3x + \frac{\mathrm{d}}{\mathrm{dx}}2y = \frac{\mathrm{d}}{\mathrm{dx}}6y + \frac{\mathrm{d}}{\mathrm{dx}}1$$

Now take the derivatives you can (those terms given in terms of x) and multiply by $\frac{\mathrm{d}y}{\mathrm{dy}}$ for those terms in terms of y.

$$\frac{\mathrm{d}}{\mathrm{dx}}3x + \frac{\mathrm{d}y}{\mathrm{dy}}\frac{\mathrm{d}}{\mathrm{dx}}2y = \frac{\mathrm{d}y}{\mathrm{dy}}\frac{\mathrm{d}}{\mathrm{dx}}6y + \frac{\mathrm{d}}{\mathrm{dx}}1$$



Implicit differentiation

$$\frac{\mathrm{d}}{\mathrm{dx}}3x+\frac{\mathrm{d}y}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{dx}}2y=\frac{\mathrm{d}y}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{dx}}6y+\frac{\mathrm{d}}{\mathrm{dx}}1$$

Rearrange derivatives as needed for differentiating the y terms

$$3+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}2y=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}6y+0$$

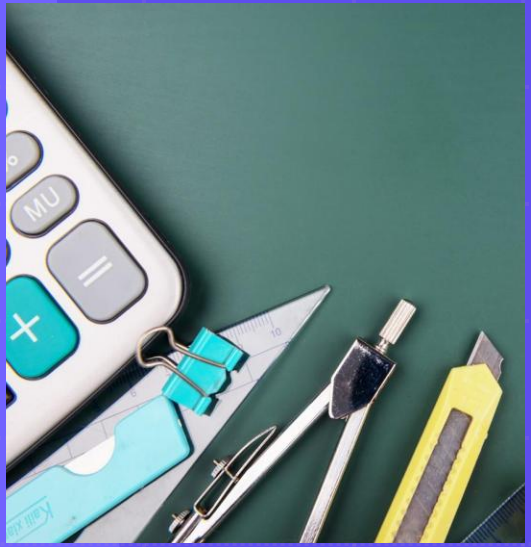
Differentiate the y terms with respect to y now

$$3+\frac{\mathrm{d}y}{\mathrm{d}x}2=\frac{\mathrm{d}y}{\mathrm{d}x}6+0$$

Rearrange and tidy up to get $\frac{\mathrm{d}y}{\mathrm{d}x}$ by itself

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{4}$$

Given this equation was a linear function, a constant (it is the gradient) is expected.



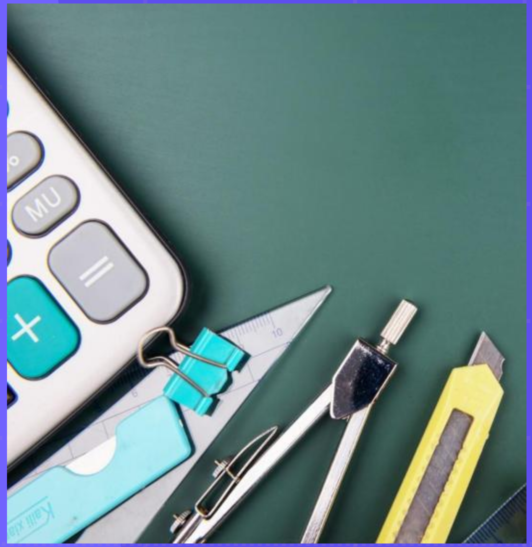
Implicit differentiation

Hints and tips here:

Markers want to see you use implicit differentiation… don't skip the steps

Look out for higher order differentiations – chain rule, product rule etc. See below.

Implicit can lurk in other question where you weren't expecting it!



Eg. Find $\frac{dy}{dx}$ in terms of x and y if $y^2 + 5x = 6x^2y$

Differentiation parametric function

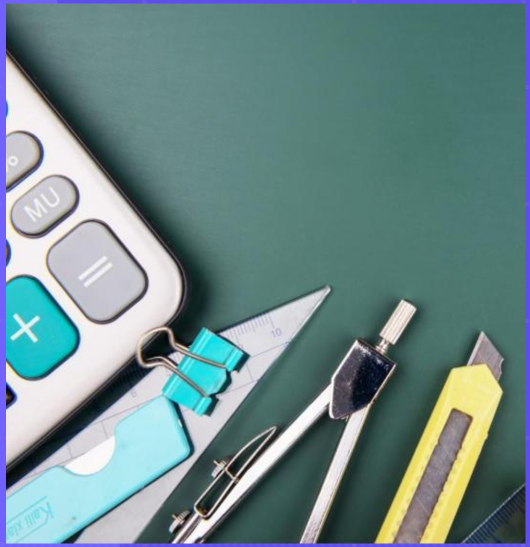
Para = two or more

Metric = to measure

To differentiate parametric function, we do not need to combine them into one function, but rather using the chain rule we can do them separately.

$$x = 3 + 2t$$
$$y = 1 - t$$

Let's determine $\frac{dy}{dx}$



Differentiation parametric function

Knowing the chain rule is

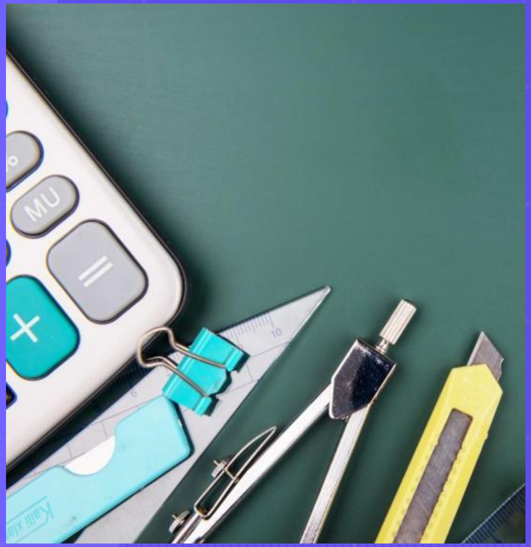
$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$x = 3 + 2t, \text{ therefore } \frac{dx}{dt} = 2$$

And $$y = 1 \ - t, \text{ therefore } \frac{dy}{dt} = -1$$

Let's determine $\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$ remembering that $\frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$

$$\frac{dx}{dy} = -1 . \frac{1}{2} = -\frac{1}{2}$$



Related rates

Related rates are being able to express the rate of change of one quantity in terms of another changing quantity.

A simplistic example could be if I know how much taller I am than Joe and how much taller Joe is than Sally, and so I can relate those two pieces of information and make a conclusion directly between myself and Sally.

If I'm blowing up a ballon, I know how much air I'm putting in (this is a rate – volume per unit time), and I know the link between volume and radius, so therefore I could determine the rate of change of the radius.

Related rates

Using the idea of a balloon we could ask if air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$. How fast is the radius of the balloon increasing when its diameter is 50 cm?

What do you know?

$$Vol = \frac{4}{3}\pi r^3$$

Diameter = 50cm → radius = 25 cm

$$\frac{\text{Change in volume}}{\text{Change in time}} = \frac{dV}{dt} = 100\,\frac{cm^3}{s}$$

Related rates

What are we trying to find?

$$\frac{\text{Change in radius}}{\text{Change in time}} = \frac{dr}{dt} = ?$$

Use the chain rule!

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$$

You will usually have to flip one of the rates, in this case we can find $\frac{dr}{dV}$ by finding $\frac{dV}{dr}$ and inverting it.

$$\frac{dV}{dr} = 4\pi r^2 \qquad \text{therefore} \qquad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

Related rates

Putting it all together

$$\frac{dr}{dt} = \frac{dV}{dt} . \frac{dr}{dV}$$

$$\frac{dr}{dt} = 100 . \frac{1}{4\pi r^2}$$

Question said we want to know the rate of change or r when the radius is 25cm

$$\frac{dr}{dt} = \frac{100}{4\pi(25)^2}$$

$$\frac{dr}{dt} \sim 0.01cm/s$$

Now remember this is the rate the radius is changing when r = 25cm. When r is less than 25 the radius is changing faster than that, and when r is greater than 25 it would be changing slower than that, $\frac{dr}{dt}$ is not a constant!

Incremental formula

In methods you would have also covered the idea that

$$\delta y \sim \frac{dy}{dx}\ \delta x$$

I won't cover anything specific here only to suggest if you need to please consult my Methods notes available on the ECU page.

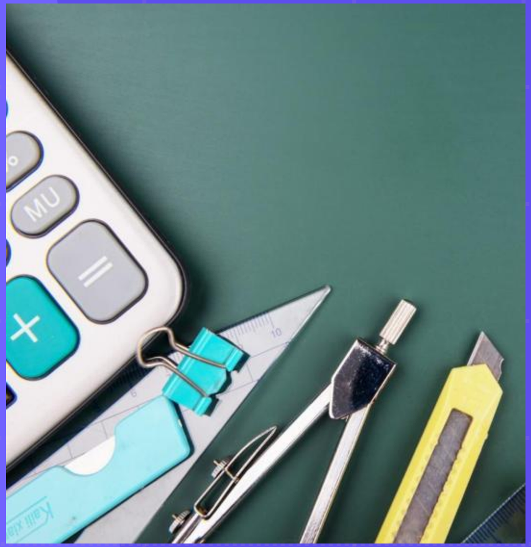
Increments formula	$\delta y \approx \frac{dy}{dx} \times \delta x$

Marginal rates

A marginal rate is simple what is the change in the variable for the addition of one more item. Given the gradient is a number which can be expressed as a fraction over 1, and that $\mathrm{m} = \frac{\mathrm{rise}}{\mathrm{run}}$, if the gradient is expressed over 1 (the addition of one more unit on the x axis) then the rise or the gradient is the increase in the y axis.

If cost is defined as $C(x)$, revenue as $R(x)$ and profit as $P(x)$, where x is the number of units sold, then $\frac{dC}{dx}, \frac{dR}{dx}$ and $\frac{dP}{dx}$ are the marginal cost, revenue and profit.

These differential equations represent marginal rates of change only when the number of units increases by 1, that is when $\delta x = 1$. Therefore, the marginal rates of change represent the cost, revenue or profit when producing one more unit.



Differential equations

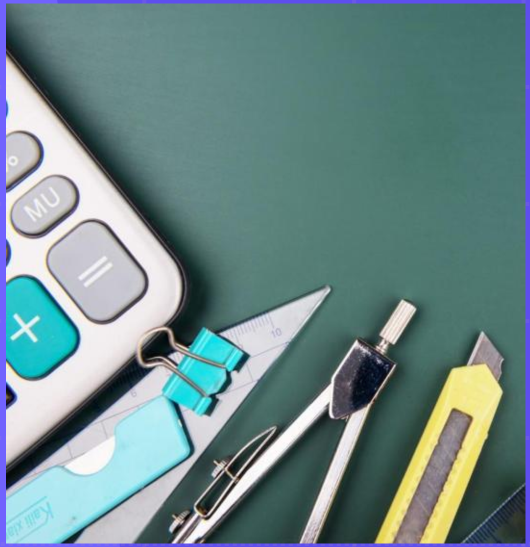
Consider the following equation:

$$\frac{dy}{dx} = x^2 + 3x + e^x$$

As this equation is a derivative, we call this type of equation a **differential equation.** It is a first order differential equation, and the syllabus contains us to these.

A differential equation gets its order from the highest derivative involved.

For example, a second derivative would be a second order differential equation.

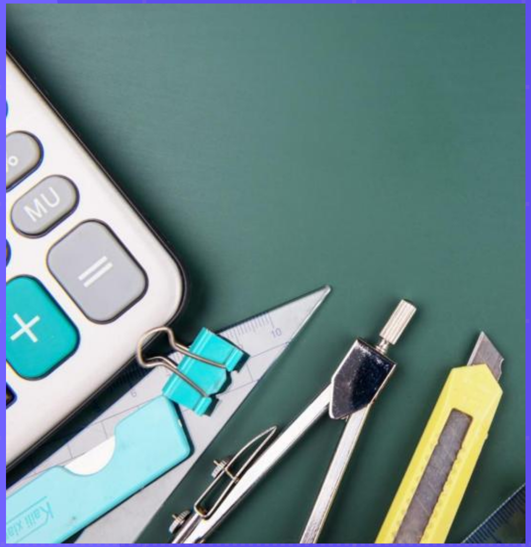


Differential equations

Thus, when asked to solve a differential equation, we are being asked to find the anti-derivative of the function involved.

For example, if $\frac{dy}{dx} = x^2 + 3x + e^x$ being asked to solve means finding $y = \int \frac{dy}{dx}\, dx$

$y = \int x^2 + 3x + e^x \, dx$ would become $y = \frac{x^3}{3} + \frac{3x^2}{2} + e^x + c$



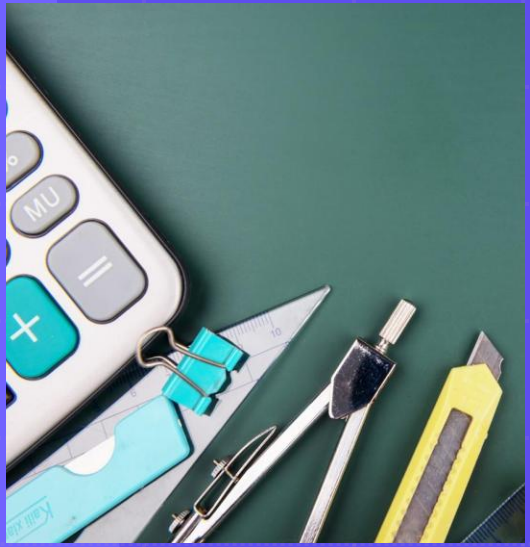
As this solution contains a '$+ c$' in it, this solution is known as the **general solution** as it is only one solution in a family of solutions. However, if we were given more information and were able to determine the '$+ c$', the solution would then be known as a **particular solution**.

Separation of variables

Now that we know how to implicitly differentiate, we could expect to be asked to solve a differential equation such as:

$$\frac{dy}{dx} = \frac{3x(x+2)}{2y-1}$$

To do this we must 'separate the variables' and integrate the x and y terms on different sides of the equation.



Separation of variables

Solve $\frac{dy}{dx} = \frac{3x(x+2)}{2y-1}$ given that when $x = 1, y = 2$

$$2y - 1\,dy = 3x(x+2)dx$$

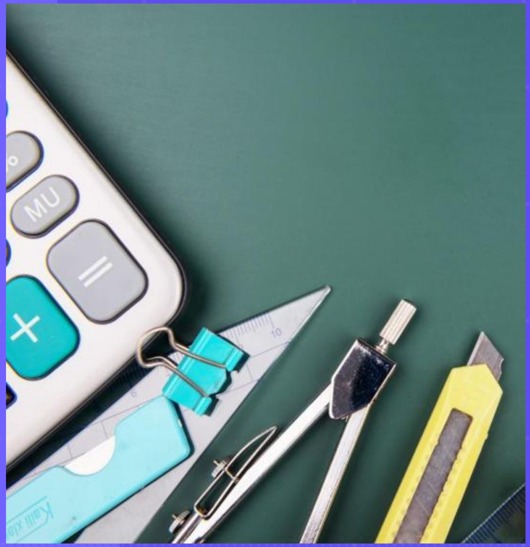
$$\int 2y - 1\,dy = \int 3x(x+2)\,dx$$

$$\int 2y - 1\,dy = \int 3x^2 + 6x\,dx$$

$$y^2 - y = x^3 + 3x^2 + c$$

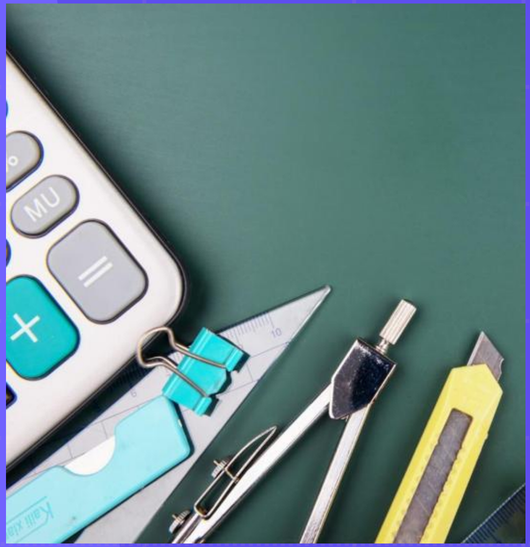
Sub $x = 1, y = 2$ and $c = -2$, hence

$$y^2 - y = x^3 + 3x^2 - 2$$



Slope fields

We're got lots of ways of working with functions and derivatives, but wait there is still more! Yes sometimes even all our tools in the tool box don't work and we need a bigger hammer!

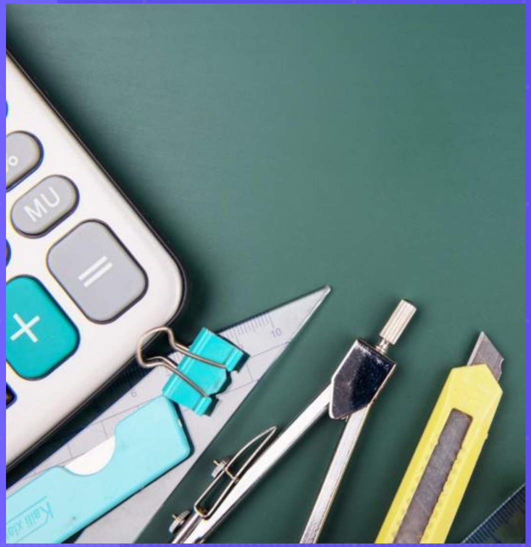


Slope fields

We now know (from Numerical Integration) that not all functions can be anti-differentiated and hence we need to find other methods of integrating them.

In the context of a differential equation, we should expect that we cannot solve all of these equations algebraically because of the same reason above.

However, if we were to plot the differential equation on a graph which we call a slope field (sometimes called a direction or gradient field), we can get a good idea of what a general solution to the differential equation will look like, and with further information we might be able to get a particular solution.



Slope fields

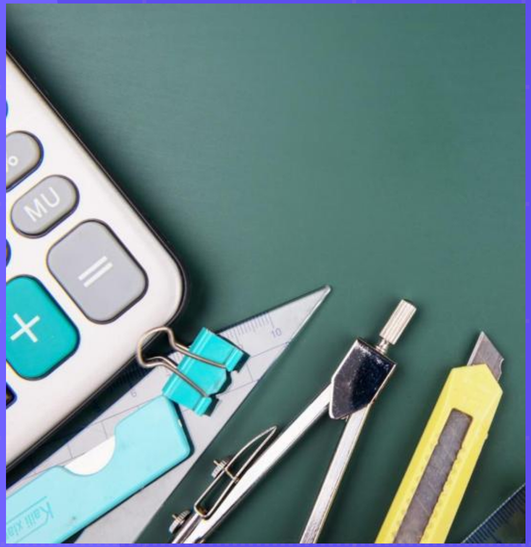
Let's say that we were asked to solve the following differential equation:

$$\frac{dy}{dx} = 3$$

We know that algebraically we can solve this differential equation (by separation of variables and then integrating each side) to give the general solution:

$$y = 3x + c$$

However, if we could not solve this differential equation algebraically, we can use a slope field to help us. Let us look at the slope field of $\frac{dy}{dx} = 3$.

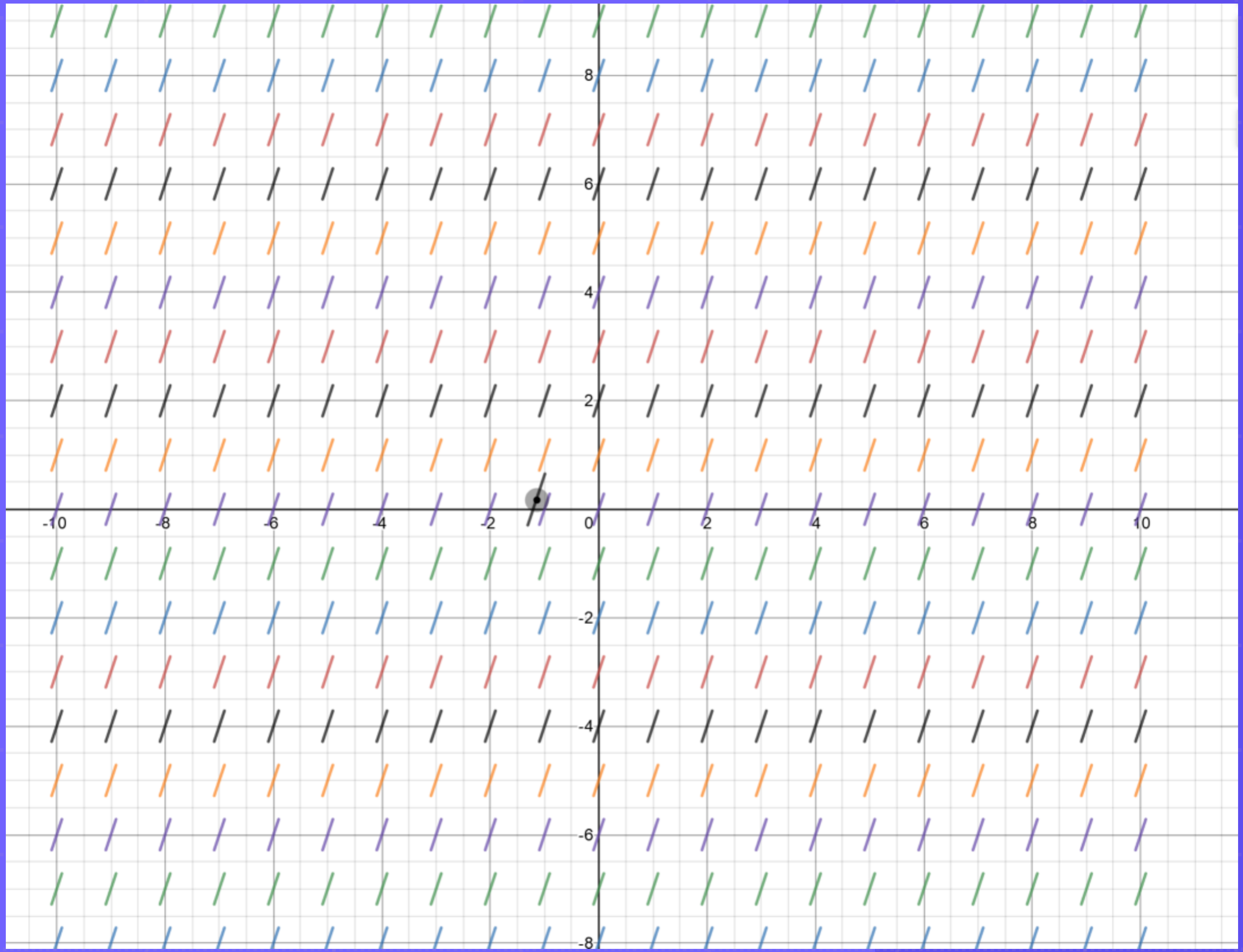


Slope fields

To create the slope field of $\frac{dy}{dx} = 3$,

we plot this gradient of 3 at each value of x

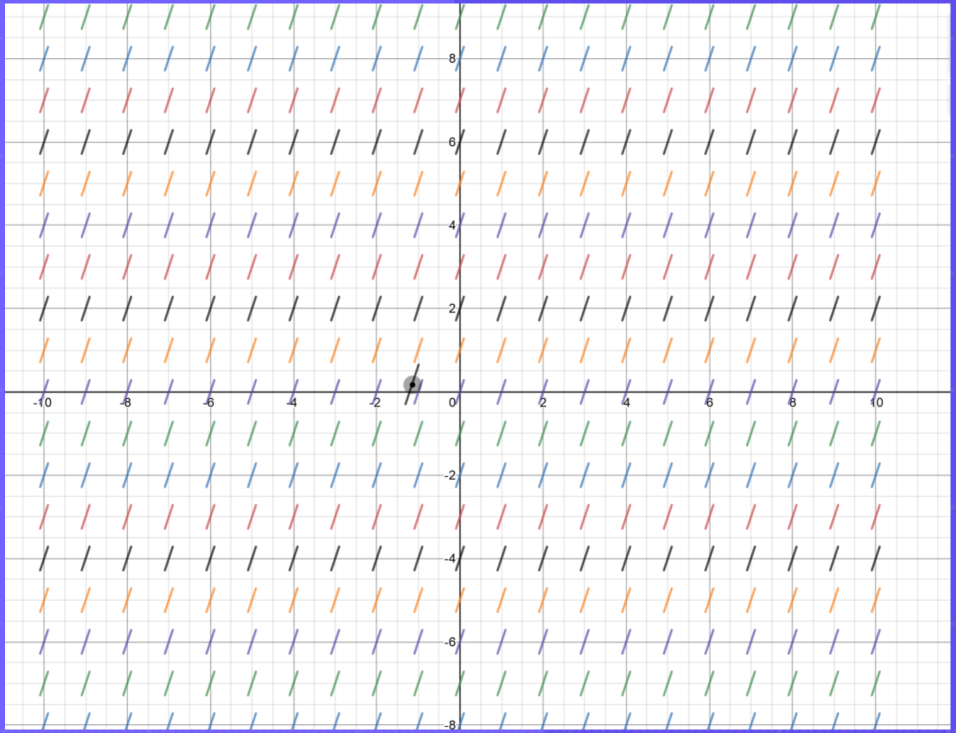
on a Cartesian plane with a small line segment



Slope fields

You can see in the slope field above, that each line segment has a gradient of 3 at each whole number for x.

From the slope field above, it should be clear that the slope field is suggesting that the solution to the differential equation, $\frac{dy}{dx} = 3$, is the family of lines that follow the equation, $y = 3x + c$.



The gradient should be quite obvious as the gradient ($\frac{dy}{dx}$) is independent of the x value. And given like most integrations we do not know the +c value, hence $y = 3x + c$.

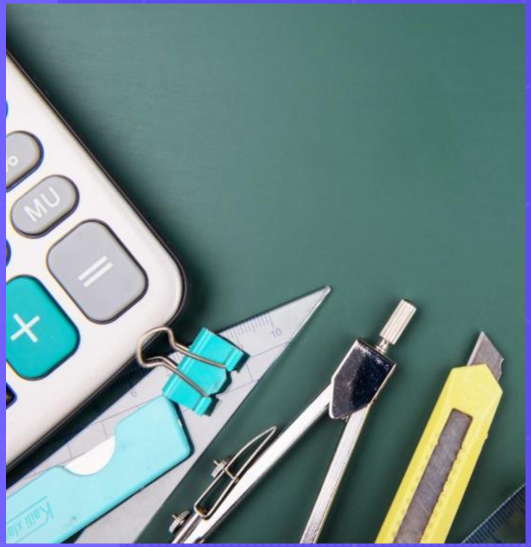
Slope fields

Let's now have a look at creating the slope field of $\frac{\mathrm{dy}}{\mathrm{dx}} = 2x$

If we determine the gradient at each whole number value of x, we would calculate the following values.

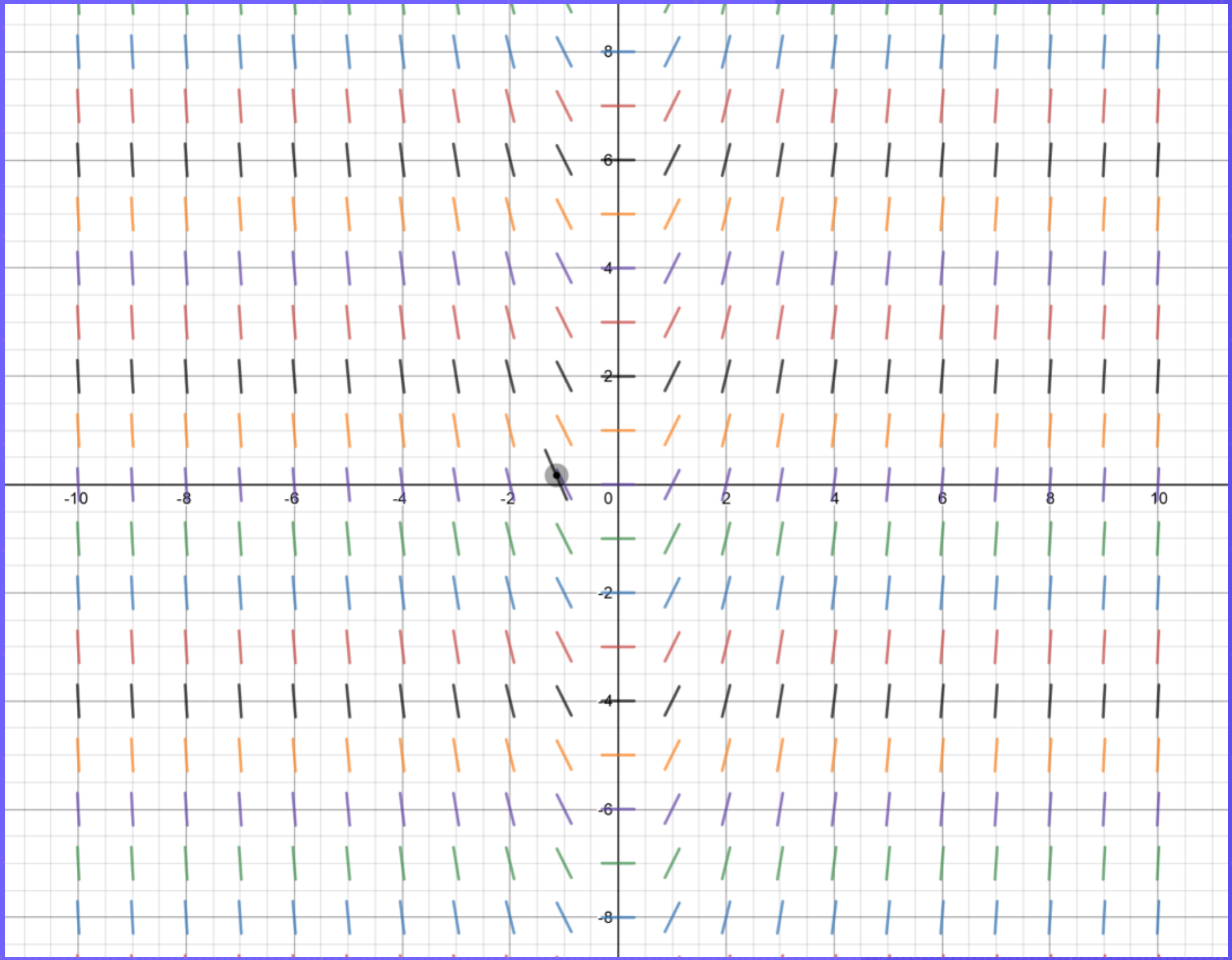
x	$\frac{dy}{dx}$
-2	-4
-1	-2
0	0
1	2
2	4
3	6

And we could generate many more values for a wider domain for a smaller interval.



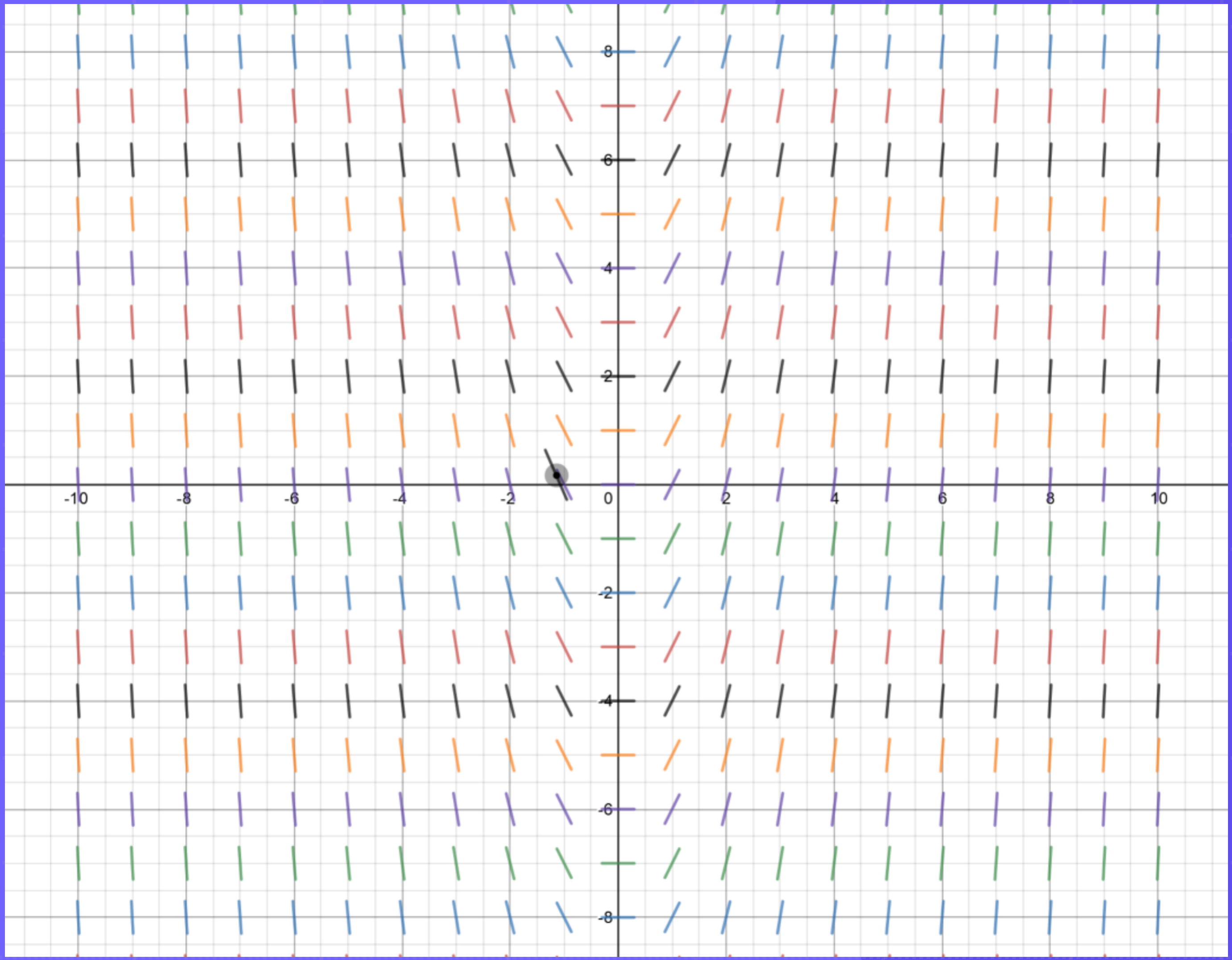
Slope fields

Hence, now if we plot these values of x and a line segment with its associated gradient we will get the following slope field



Slope fields

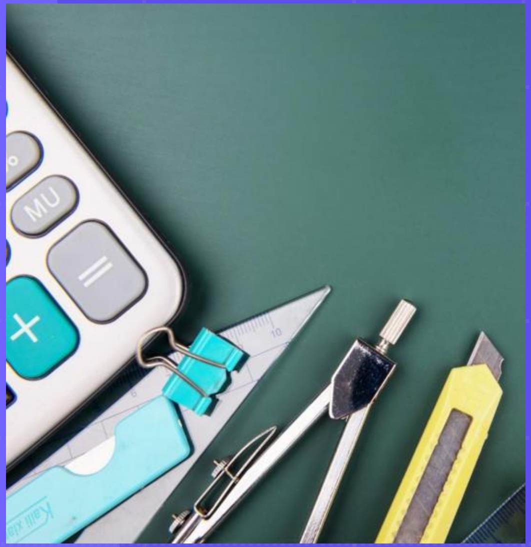
You should be able to see again that this slope field is suggesting that the solution to differential equation $\frac{dy}{dx} = 2x$, is the general solution $y = x^2 + c$, which is what you would expect if you solved the differential equation algebraically.



Differential equations

If $\frac{dP}{dt} = kP,$ then $\mathrm{P} = P_0 e^{kt}$ which you may (should!!) recall from Methods, commonly referred to as exponential growth and decay.

Now that we know how to solve differential equations by separating the variables, we can now solve the differential equation above and arrive at an equation for P through integration.



Differential equations

Figures indicate that a particular country's instantaneous growth rate is always approximately 5% of the population at that time.

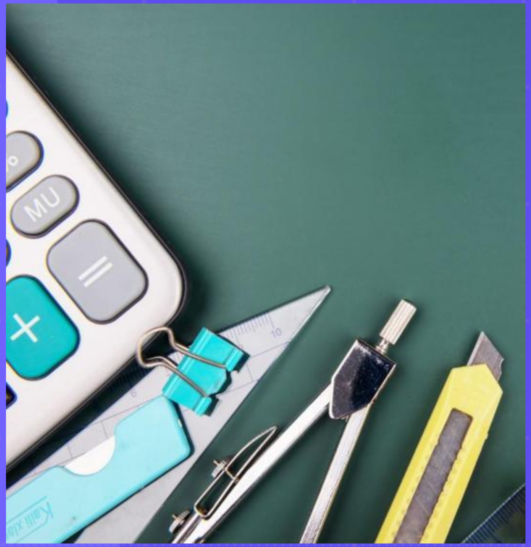
(a) How long does it take the population to double?

(b) If the population now is 2 000 000 in how many years will it be 8 000 000?

That means $\frac{dP}{dt} = kP$ becomes $\frac{dP}{dt} = 0.05P$

And now separating variables becomes

$$\frac{1}{P}dP = 0.05\ \mathrm{dt}$$



Differential equations

$$\frac{1}{P}dP = 0.05\ \mathrm{dt}$$

$$\int \frac{1}{P}dP = \int 0.05\ \mathrm{dt}$$

$$lnP = 0.05t + c$$

Raise both sides to 'e'

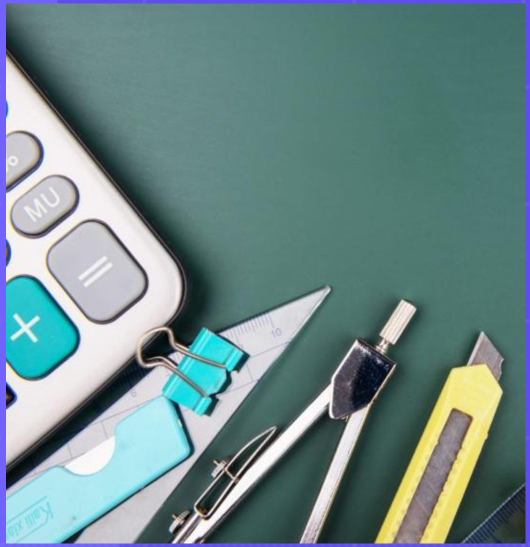
$$e^{lnP} = e^{0.05t+c}$$

$e^{lnP} = P$, $e^{0.05t+c} = e^{0.05t}.e^{c}$

$$P = e^{0.05t}.e^{c}$$

$e^{c} = P_0$, since when $t = 0$, $e^{0.05t} = e^{0} = 1$, hence $P_0 = e^{c}$

$$P = P_0e^{0.05t}$$



Differential equations

Figures indicate that a particular country's instantaneous growth rate is always approximately 5% of the population at that time.

$$P = P_0 e^{0.05t}$$

(a) How long does it take the population to double?

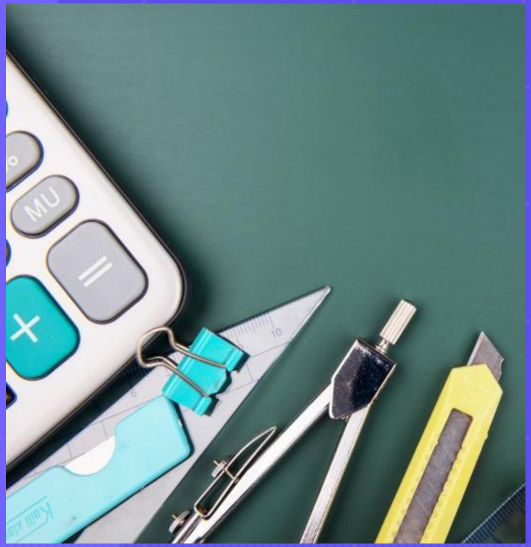
We're looking for $P = 2P_0$, hence, $2P_0 = P_0 e^{0.05t}$ or $2 = e^{0.05t}$

(b) If the population now is 2 000 000 in how many years will it be 8 000 000?

Now we're wanting to start at 2,000,000 ie P_0 and finish at 8,000,000 ie P

Hence $P = P_0 e^{0.05t}$ becomes $8{,}000{,}000 = 2{,}000{,}000 e^{0.05t}$ which becomes

$4 = e^{0.05t}$, take ln both sides, $\ln 4 = 0.05t \ln e$, and $\ln e = 1$ therefore $t = \dfrac{ln4}{0.05}$



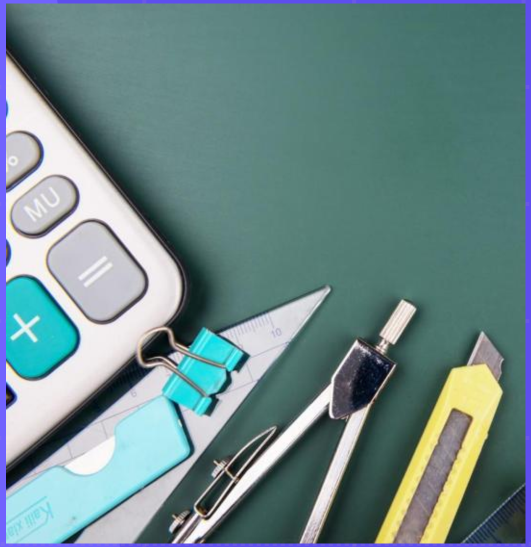
Logistics equations

We've just seen $P = 1500 \times 1.11^t$ and $\mathrm{P} = P_0 e^{kt}$ for modelling population growth…

But…. David Suzuki said, "Infinite growth in a finite world is impossible".. So, we have an impass!

These models above assume that the population would increase indefinitely.

In real life though, access to resources e.g. food, water etc. as well as predators (for animals) would mean that the population growth would start to slow down at some point and eventually the population growth would 'level out'.

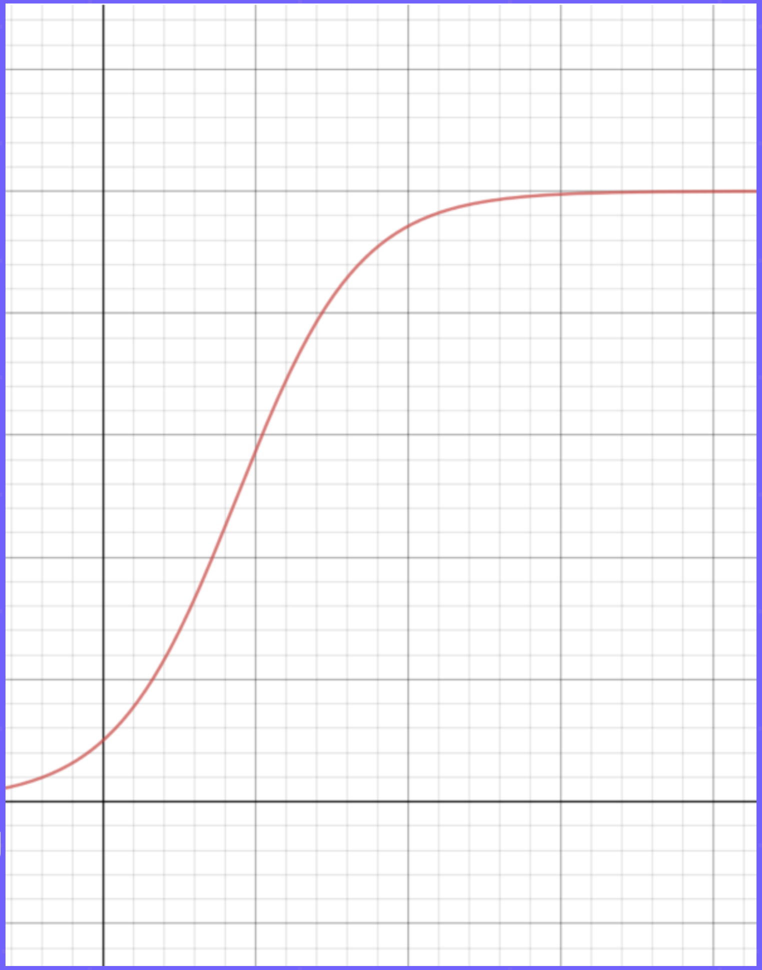


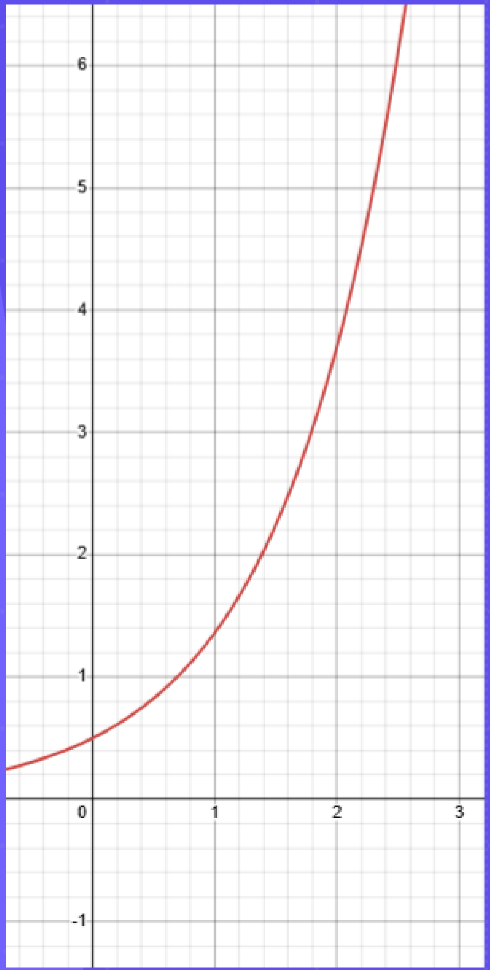
Logistics equations

From a graphical point of view, population that is modelled in this way may be more suitable for population growth than one that assumes uninhibited growth.

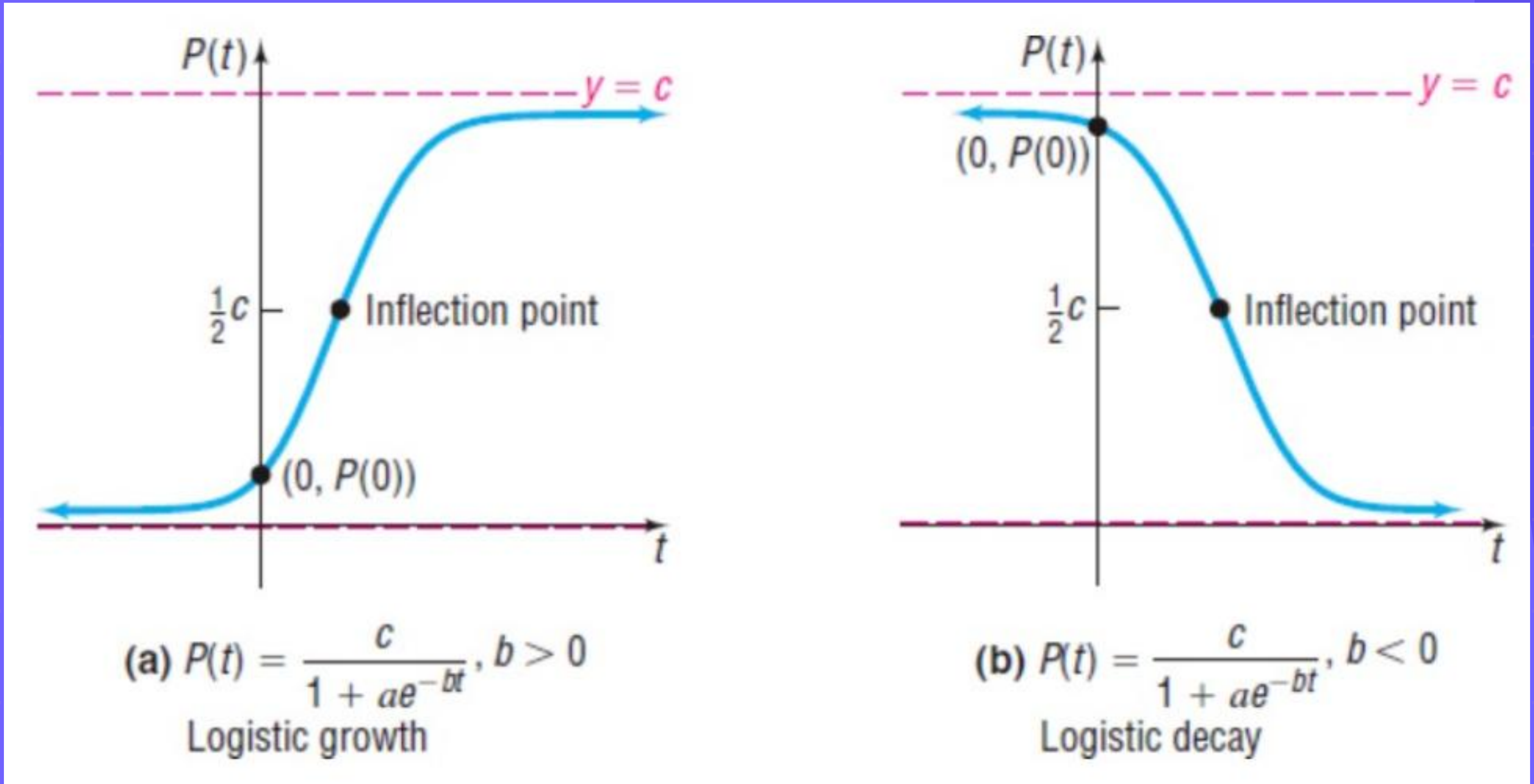
These are called a S-shaped or sigmoidal curve

- Initially, growth is nearly exponential when P << K
- Growth slows as P approaches K due to
- environmental limits.
- Eventually, the population stabilizes at the carrying
- capacity K.





Logistics equations



Logistics equations

Equations are usually presented in two ways for logistics equations

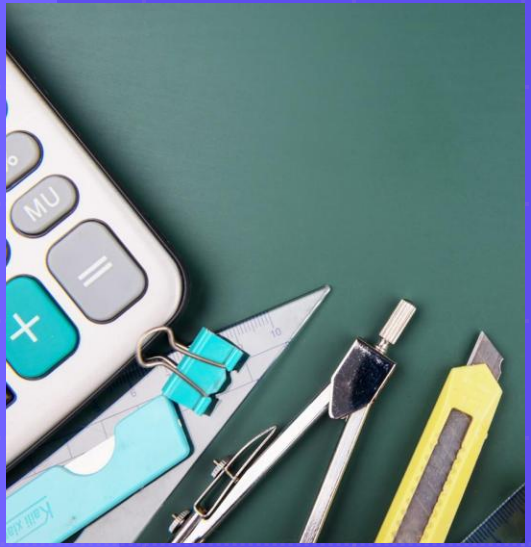
$$\frac{dP}{dt} = 0.0001P(5000 - P) \qquad \text{or} \qquad \frac{dN}{dt} = \frac{N}{4} - \frac{N^2}{8000}$$

Both are expressing the same information to us and can be converted into

$$P = \frac{\mathrm{c}}{1+\mathrm{a}e^{-\mathrm{b}t}} \ ,$$

if $b > 0$ then logistic growth, if $b < 0$ then logistic decay.

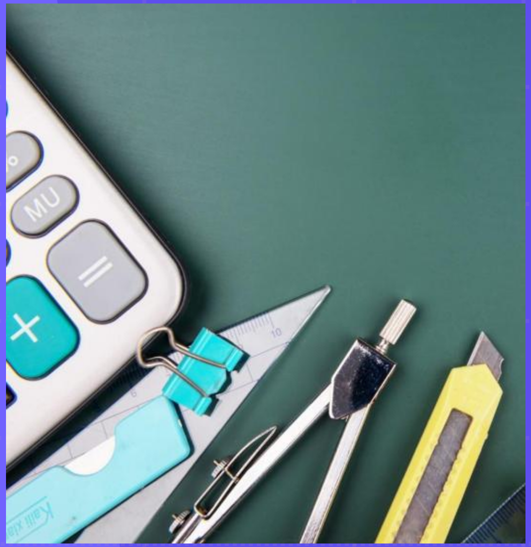
I doubt you will need to derive the above equations into this form, but it can be done and hopefully your teachers showed you all the way through.



Logistics equations

And our formula sheet gives us

Growth and decay	
Exponential equation	$\frac{dP}{dt} = kP \Leftrightarrow P = P_0 e^{kt}$
Logistic equation	$\frac{dP}{dt} = rP(k-P) \Leftrightarrow P = \frac{kP_0}{P_0 + (k-P_0)e^{-rkt}}$



Modelling motion – rectilinear motion

So far you have learnt in rectilinear motion that the displacement, velocity and acceleration are given in terms of <u>time</u>.

For example, if the displacement of a particle was given as

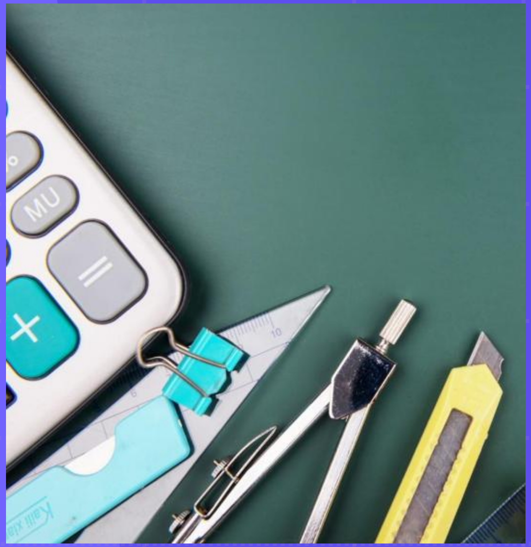
$$x = 10t - t^2$$

Then its velocity (by differentiation) would be given as

$$v = \frac{dx}{dt} = 10 - 2t$$

And its acceleration (by differentiation a second time) would be given as

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -2$$



Modelling motion – rectilinear motion

Our formula sheet gives us

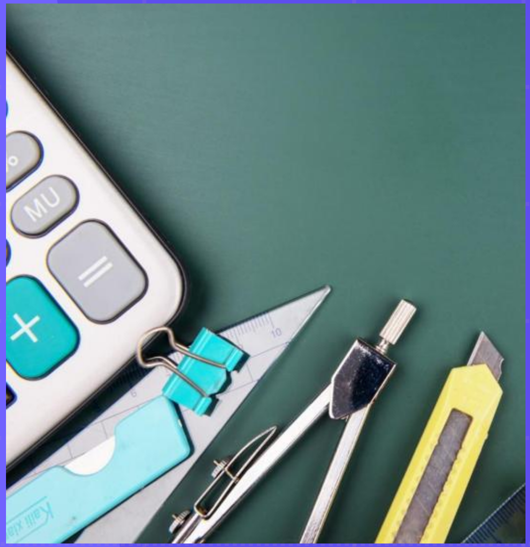
Acceleration	$\dfrac{dv}{dt}$ or $v\dfrac{dv}{dx}$ or $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

These are the three ways acceleration can be expressed.

In Specialist we must be able to work with functions of velocity and acceleration that are given in <u>terms of the displacement and not time</u>. For example, we might be given that the velocity of a particle at time t, is given by

$v = 3x$, where x is the particle's displacement at time, t.

We could then be asked to find the particle's acceleration or its displacement given some initial conditions.



Modelling motion – in terms of time

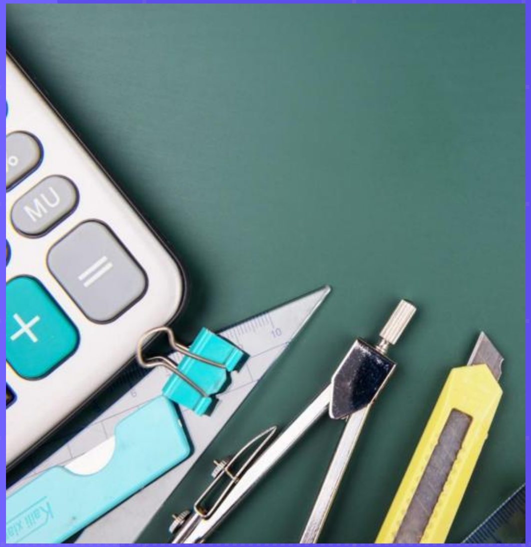
A particle is initially at an origin O. It is projected away from O and moves in a straight line such that its displacement from O, t seconds later, is x metres where

$$x = t(10 - t).$$

(a) Find the speed of the particle when $t = 12$

(b) Find the value of t when the particle comes to rest and the distance from the origin at that time.

(c) Find the distance the particle travels from $t = 3$ to $t = 6$.



Modelling motion – in terms of velocity

If $v = 3x$ find the acceleration when $x = 2$.

$$v = \frac{\mathrm{dx}}{\mathrm{dt}} = 3x \quad \text{and} \quad \mathrm{a} = \frac{\mathrm{dv}}{\mathrm{dt}} \quad \text{and} \quad \mathrm{a} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{dx}{dt}$$

So hence,

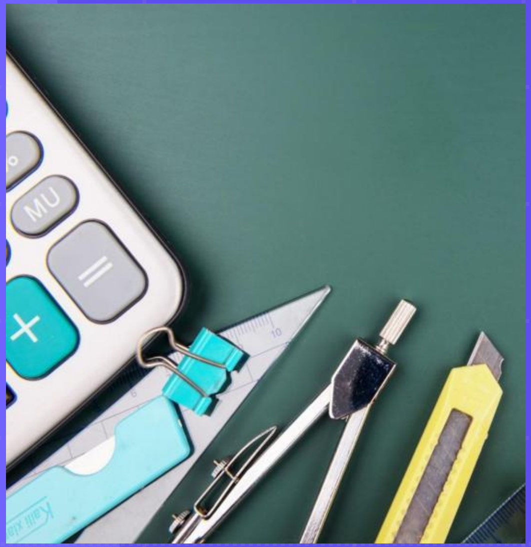
$$\mathrm{a} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{dx}{dt} \text{ and from the question } v = 3x \text{ meaning } \frac{\mathrm{d}v}{\mathrm{d}x} = 3$$

$$\mathrm{a} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{dx}{dt}$$

$$\mathrm{a} = (3)(3x)$$

$$\mathrm{a}(2) = (3)(3)(2)$$

$$\mathrm{a} = 18\, ms^{-1}$$



Simple Harmonic Motion (SHM)

You're never going to be as cool as this guy →

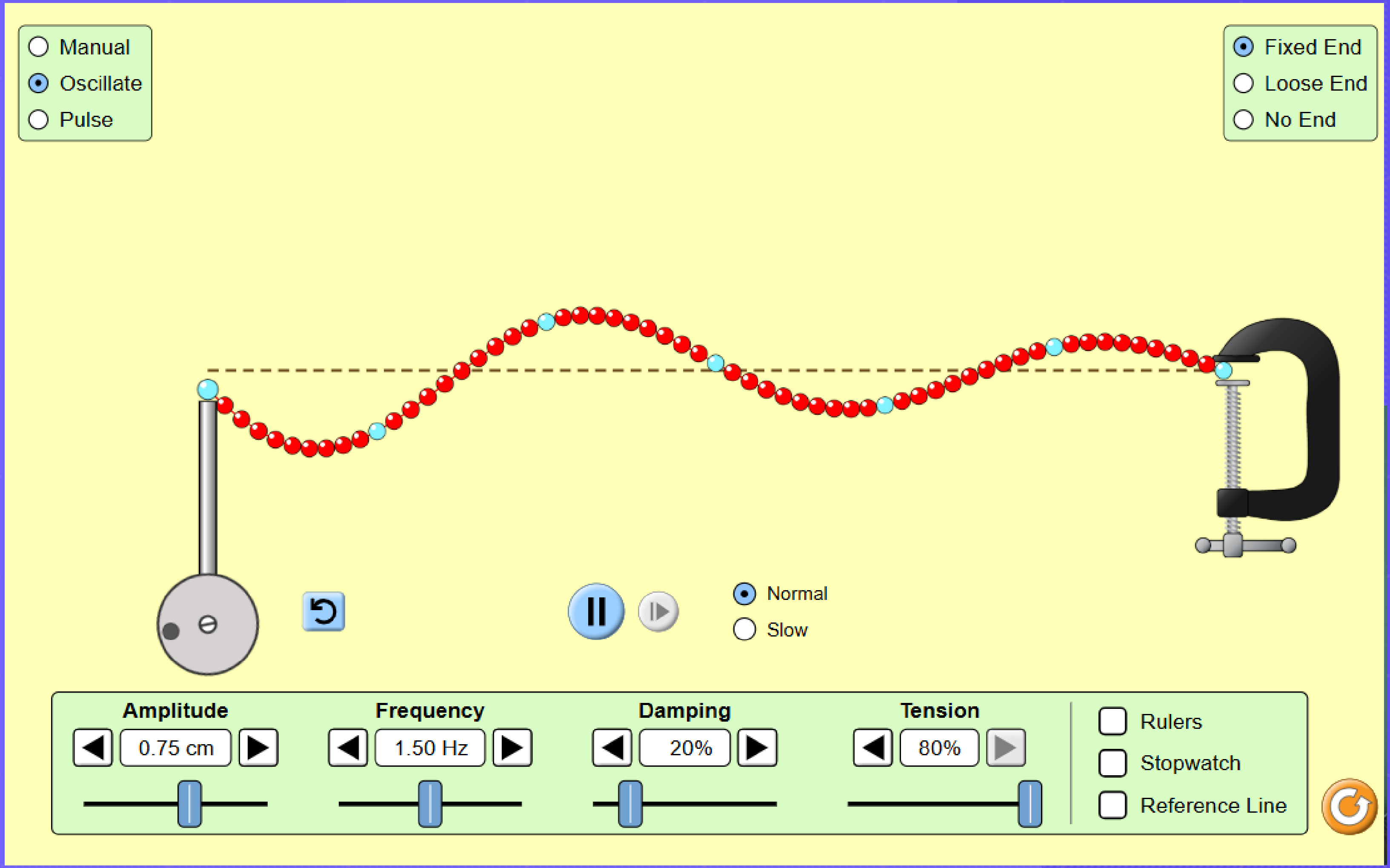
The motion that you would experience on the swing would be a back-and-forth motion. In more technical terms you would be **oscillating** around the point that you first started at. We call this type of motion **simple harmonic motion.**

Other common occurrences of SHM are an oscillating spring, a simple pendulum or circular motion.

Simple Harmonic Motion (SHM)

Modelling SHM with a string

[Wave on a String]



Simple Harmonic Motion (SHM)

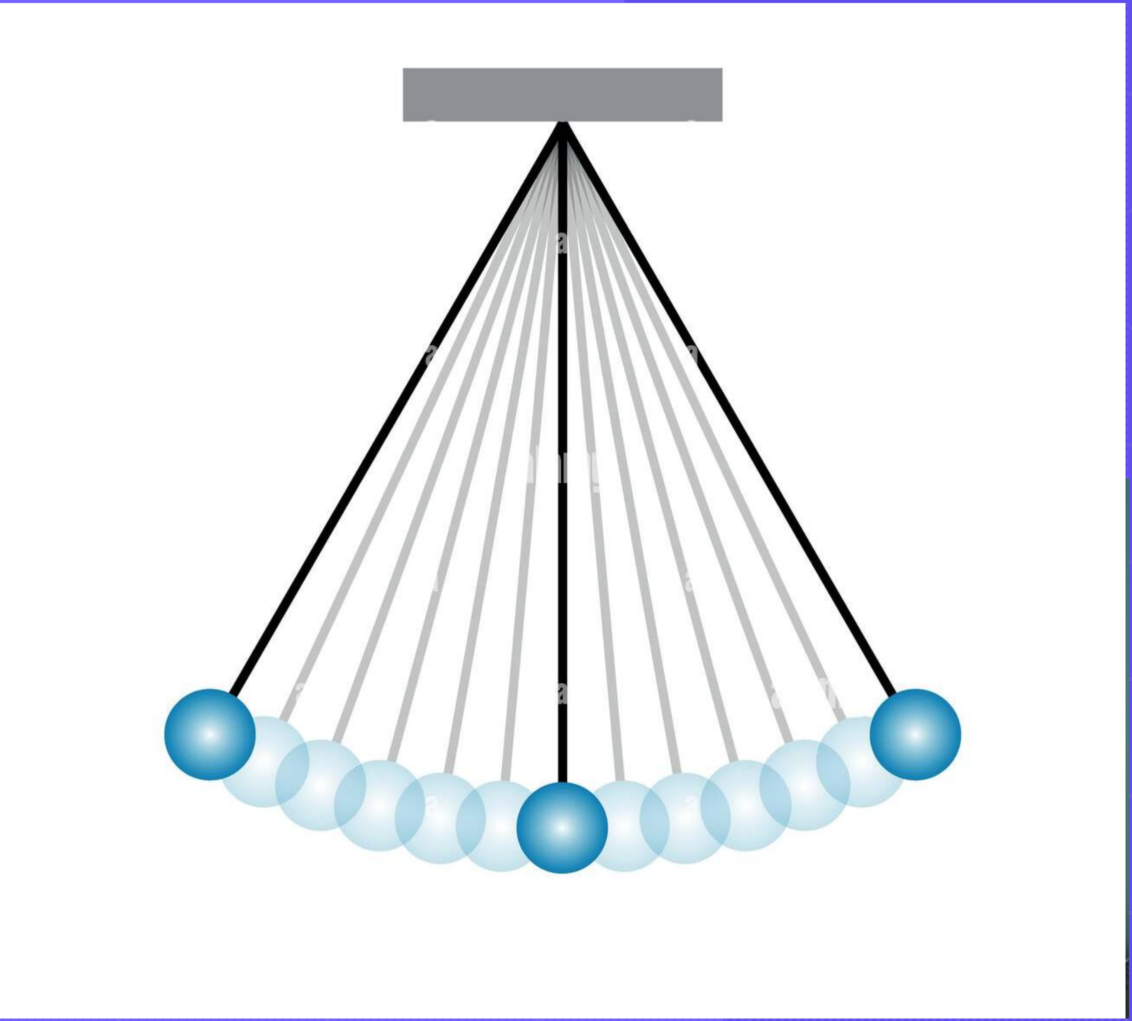
The origin is called the mean position.

As it moves to the right from O

- its velocity is in the direction of movement
- its acceleration is towards the origin
- its velocity is decreasing

At the outer most right-hand side point

- its displacement is a maximum
- its velocity is zero
- its acceleration remains towards the origin



Simple Harmonic Motion (SHM)

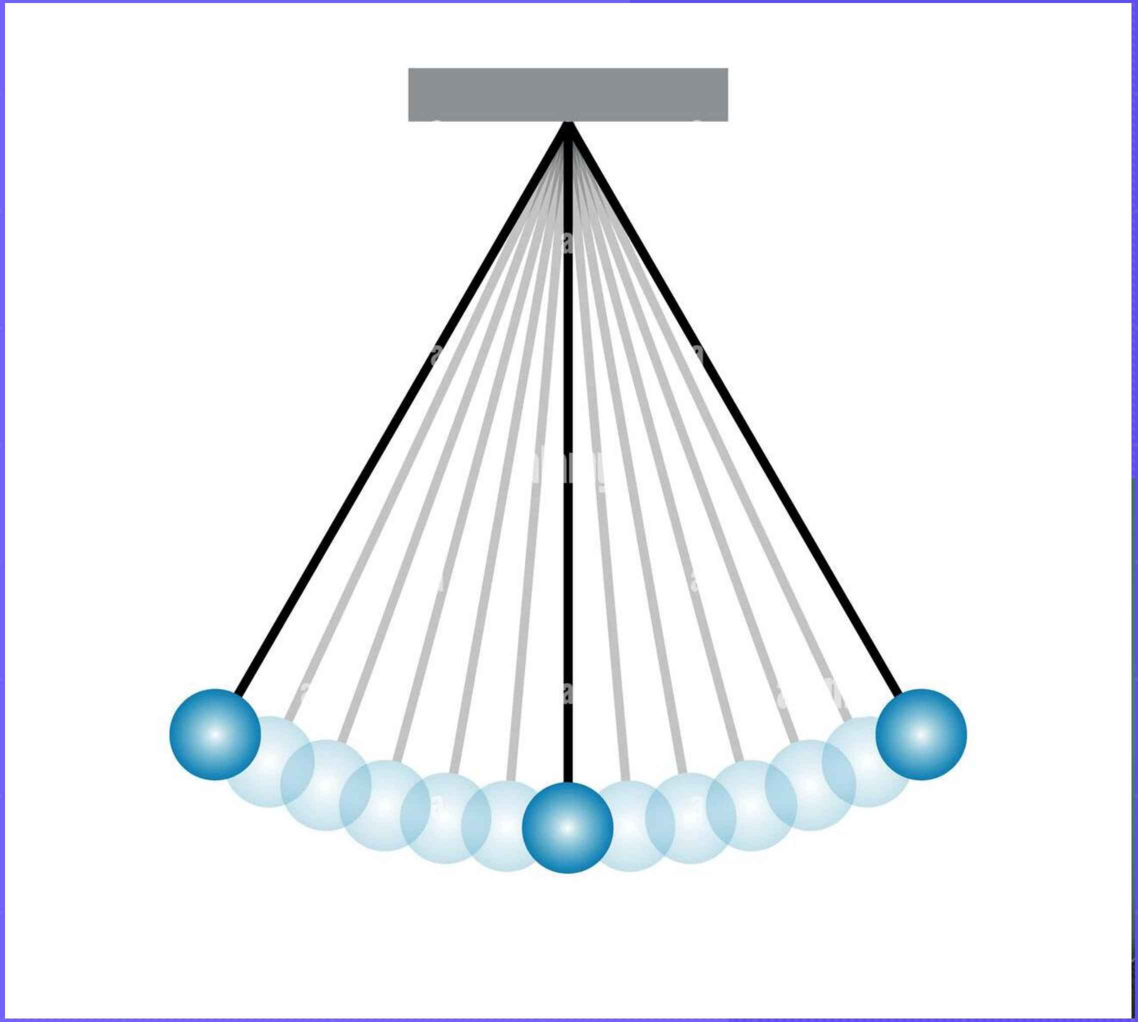
The origin is called the mean position.

As it moves from the outermost point towards to origin

- its velocity is in the direction of movement
- its acceleration is towards the origin
- its velocity is increasing

As it passes the origin

- its displacement is zero
- its velocity is at a maximum
- its acceleration (momentarily) is zero



Simple Harmonic Motion (SHM)

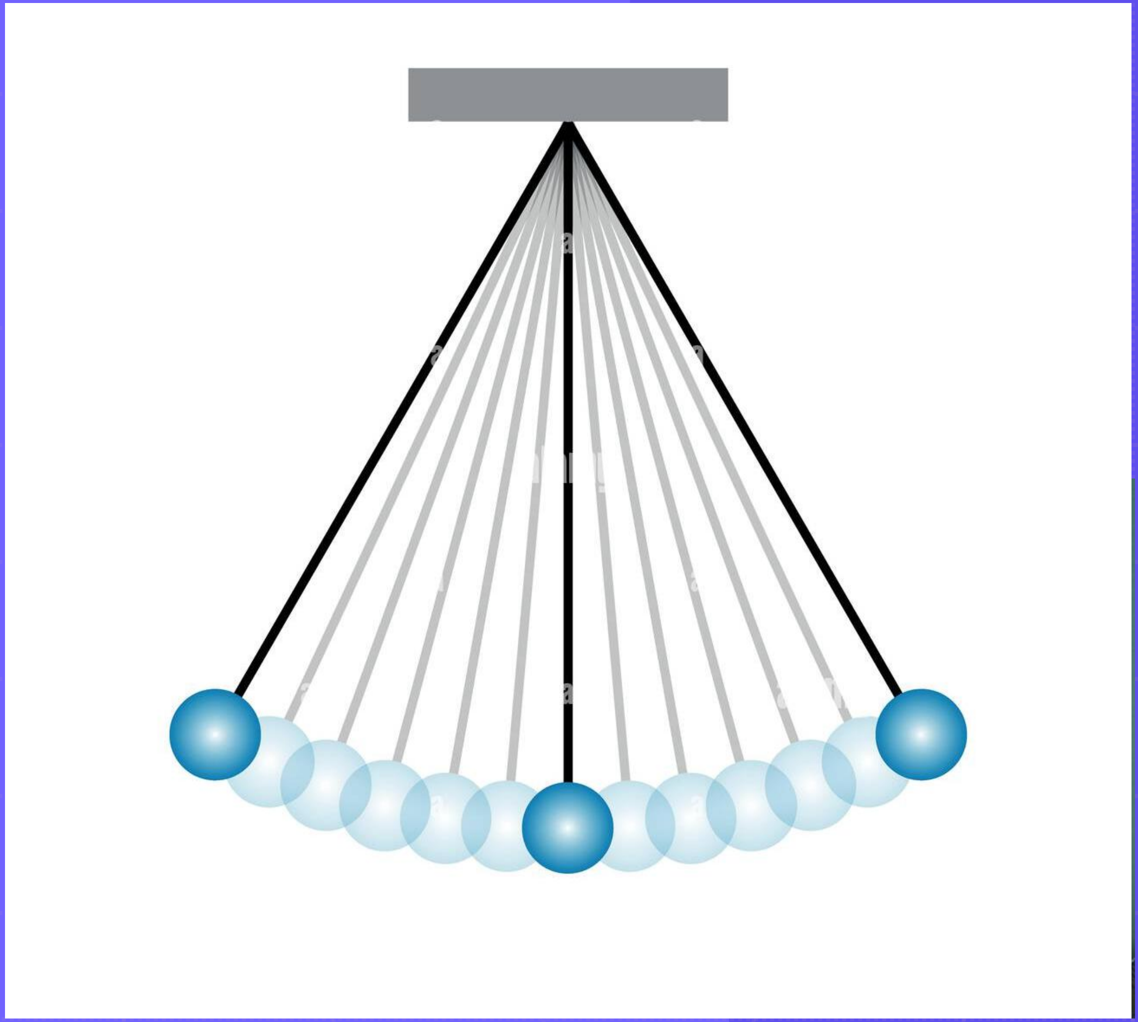
The origin is called the mean position.

As it moves to the left from O

- its velocity is in the direction of movement
- its acceleration is towards the origin
- its velocity is decreasing

At the outer most left-hand side point

- its displacement is a maximum
- its velocity is zero
- its acceleration is towards the origin



Simple Harmonic Motion (SHM)

In this situation the acceleration is proportional to its displacement.

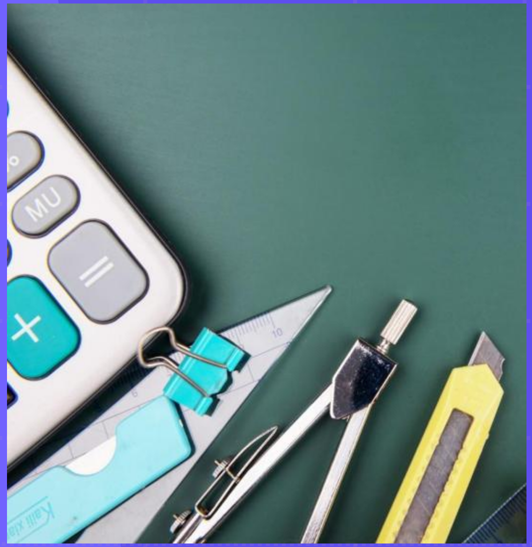
$$\frac{d^2x}{dt^2} = -k^2x$$

where k is a constant.

This is more commonly written as:

$$\ddot{x} = -k^2x$$

where $\ddot{x}$ symbolises the second derivative of the displacement (which would be acceleration)



Simple Harmonic Motion (SHM)

Starting with this expression we can derive an expression for velocity

$$\ddot{x} = -k^2 x$$

$$\frac{dv}{dt} = -k^2 x$$

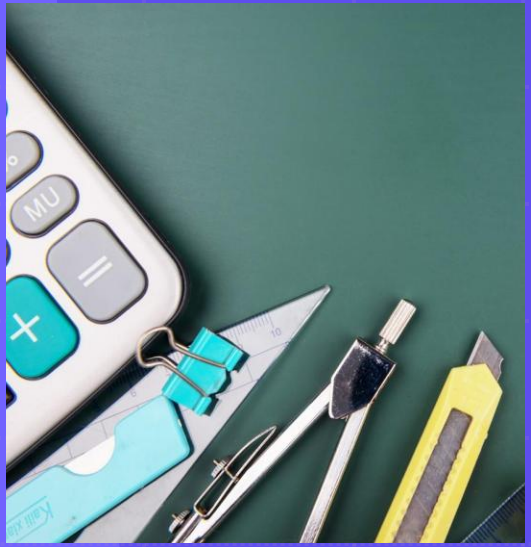
$$\frac{dv}{dx}\frac{dx}{dt} = -k^2 x$$

Remember that $\mathrm{v} = \frac{dx}{dt}$ hence

$$v\frac{dv}{dx} = -k^2 x$$

Separate your variables and add the integral sign

$$\int v\ dv = \int -k^2 x\ \mathrm{dx}$$



Simple Harmonic Motion (SHM)

$$\int v\, dv = \int -k^2 x \text{ dx}$$

$$\frac{(v^2)}{2} = \frac{-k^2x^2}{2} + c$$

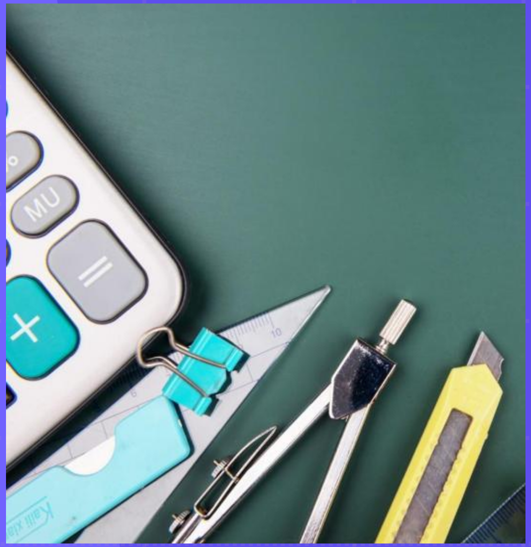
Now let the amplitude $= a$, hence when $v = 0, x = a$. Let's sub that in

$$\frac{(0^2)}{2} = \frac{-k^2a^2}{2} + c$$

$$0 = \frac{-k^2a^2}{2} + c$$

$$c = \frac{k^2a^2}{2}$$

Let's now sub our c value back in above



Simple Harmonic Motion (SHM)

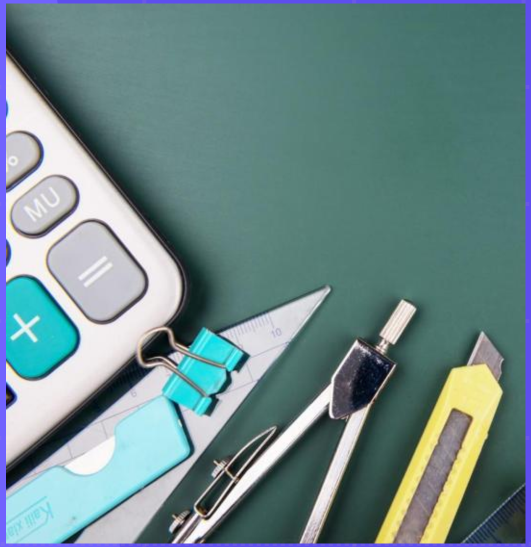
$\frac{(v^2)}{2} = \frac{-k^2x^2}{2} + c$, and $c = \frac{k^2a^2}{2}$

$$\frac{(v^2)}{2} = \frac{-k^2x^2}{2} + \frac{k^2a^2}{2}$$

Cancel the 2 and take $-k^2$ as a common factor out front

$$v^2 = -k^2(x^2 - a^2)$$

Where $v = velocity, a = acceleration$ and $x = displacement.$ k is a constant based on the system.



Simple Harmonic Motion (SHM)

Period

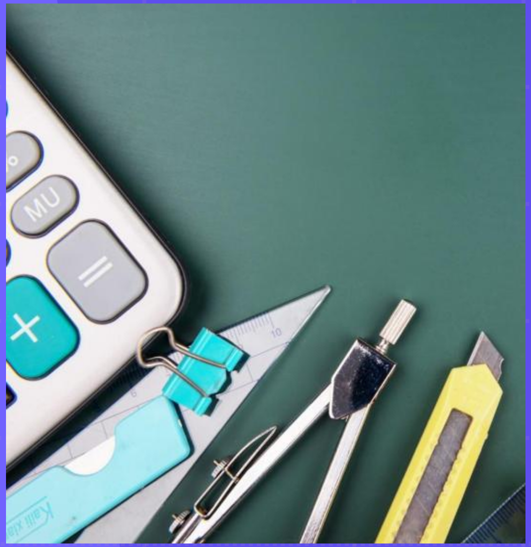
As SHM is periodic, that is, it crosses over the same point again and again, we should expect to be able to find a time period in which is does its oscillation. Hence the period represents the amount of time it takes to return to the same point travelling from the original direction.

The period of a body in SHM is given by $Period = \frac{2\pi}{k}$

Phase angle

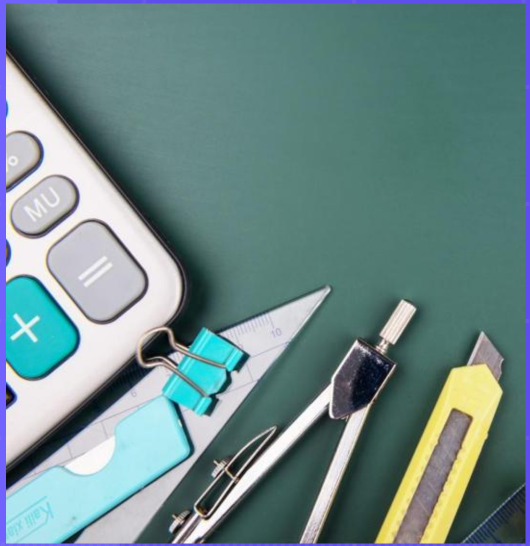
The constant α in the equation $x = a\sin(kt + \alpha)$ is known as the phase angle and is dependent on the position of the body when time commences i.e. when $t = 0$.

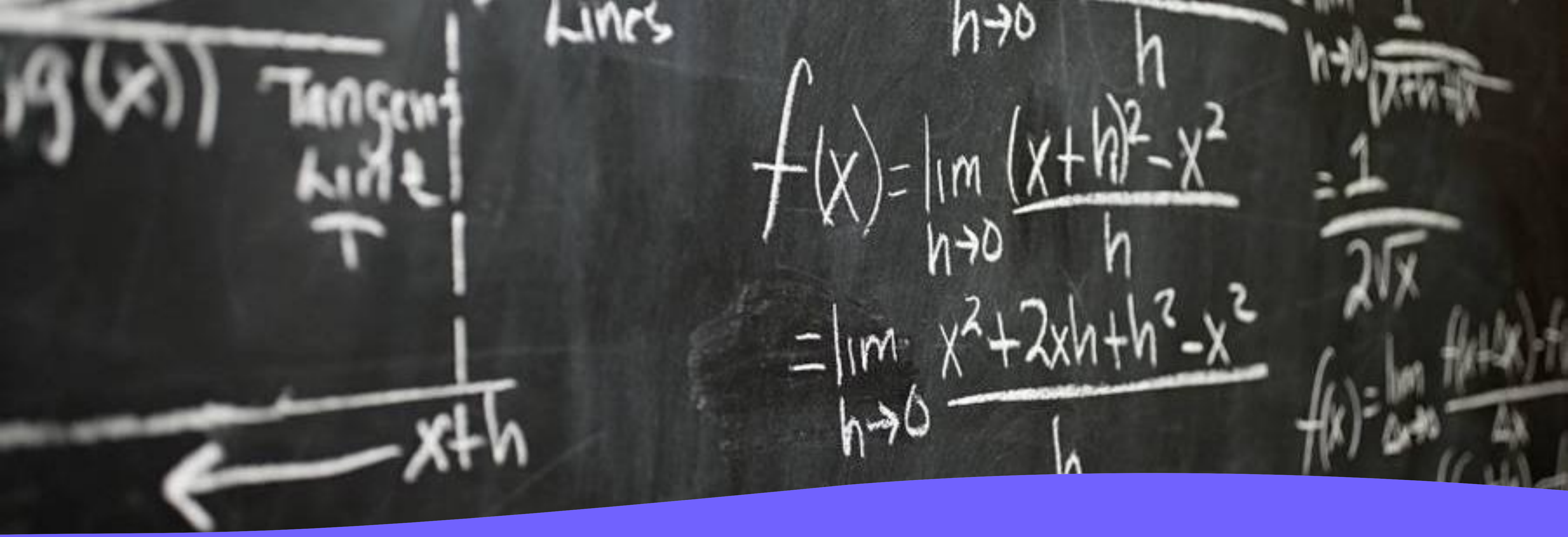
Initially at position O, $\alpha = 0$, initial position at one of the extreme points, $\alpha = \frac{\pi}{2}$



Simple Harmonic Motion (SHM)

Simple harmonic motion
If $\dfrac{d^2x}{dt^2} = -k^2x$ then $x = A\sin(kt + \alpha)$ or $x = A\cos(kt + \beta)$ where A is the amplitude, α and β are phase angles, v is the velocity and x is the displacement
$v^2 = k^2(A^2 - x^2)$ Period: $T = \dfrac{2\pi}{k}$ Frequency: $f = \dfrac{1}{T}$





Questions?

WACE 2022 Question 11 (C)

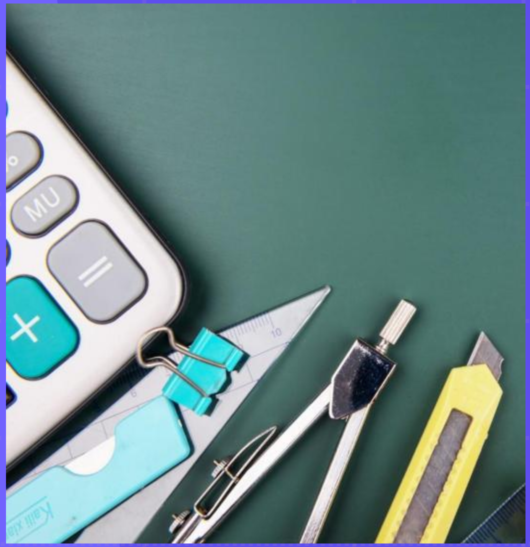
Question 11 **(8 marks)**

A Formula One (F1) racing car has an initial displacement of 192 metres with an initial velocity of 24 metres per second. It accelerates for a period of 11 seconds in a straight line so that its velocity v metres per second and displacement x metres are related by the equation:

$$v(x) = \frac{x}{8}$$

(a) Determine the acceleration a as a function of displacement x i.e. determine $a(x)$.

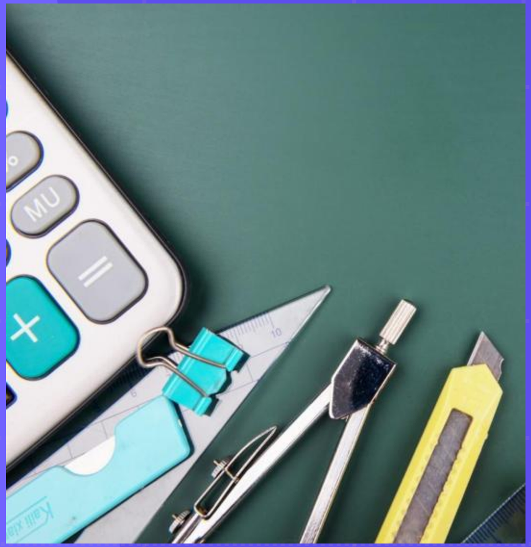
(2 marks)



WACE 2022 Question 11 (C)

(b) Determine the displacement x as a function of time t. (2 marks)

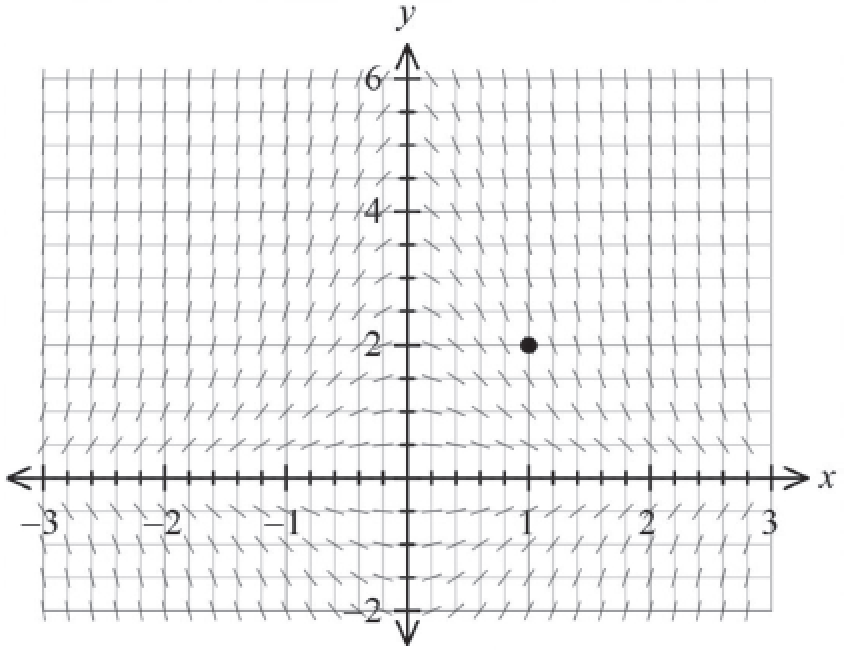
(c) Calculate the top speed reached and the distance travelled during the 11 second period of acceleration. (4 marks)



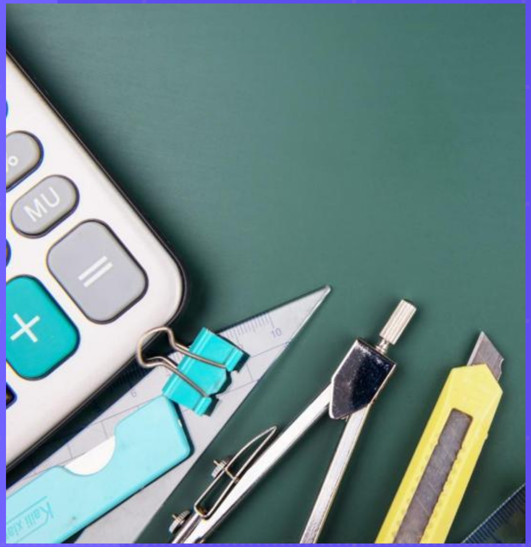
WACE 2023 Question 11 (C)

Question 11 **(5 marks)**

A slope field is given by the equation $\frac{dy}{dx} = k(xy)$ where k is a constant.

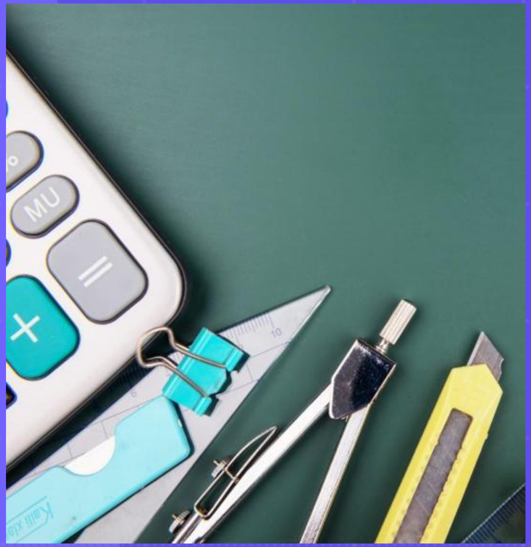


(a) The value of the slope field at the point $(1, 2)$ is equal to -4. Determine the value of the constant k. (2 marks)



WACE 2023 Question 11 (C)

(b) Determine the equation for the solution curve that contains the point $(1, 2)$ and draw this curve on the diagram above. (3 marks)



Unit 4 Topic 3

Statistical inference

4.3.1 – 4.3.7

Syllabus points

Sample means

4.3.1 examine the concept of the sample mean X as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ

4.3.2 simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate properties of the distribution of $\overline{X}$ across samples of a fixed size *n* , including its mean μ its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X), and its approximate normality if n is large

4.3.3 simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate the approximate standard normality of $\frac{\overline{X}-\sigma}{\frac{s}{\sqrt{n}}}$ for large samples (n>=30) where *s* is the sample standard deviation

Syllabus points

Confidence intervals for means

4.3.4 examine the concept of an interval estimate for a parameter associated with a random variable

4.3.5 use the approximate confidence interval $\bar{x} - \frac{zs}{\sqrt{n}}, \bar{x} + \frac{zs}{\sqrt{n}}$ as an interval estimate for the population mean μ, where z is the appropriate quantile for the standard normal distribution

4.3.6 use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain μ

4.3.7 use $\bar{x}$ and s to estimate μ and σ to obtain approximate intervals covering desired proportions of values of a normal random variable, and compare with an approximate confidence interval for μ

Sampling

Population – the entire group upon which data can be collected. Usually, we'd be talking about the entire school, club, state, country etc. The population can be defined to suit the sampling needed.

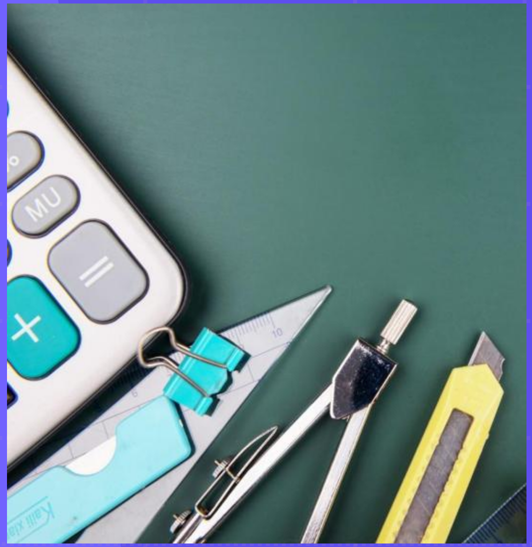
Census – To collect data from the entire population

Sample – To collect data from a subset of the population

Fair sampling – this is the million-dollar question!

A fair sample must represent the entire population and ensure that every data point in the population has an equal chance of being included.

We often conduct random sampling to <u>attempt</u> to achieve equal chance

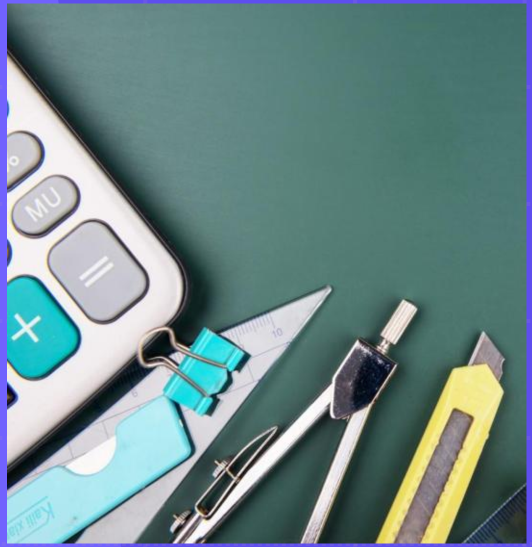


Central limit theorem (CLT)

CLT states that, under certain conditions, the distribution of the sample mean of a large number of independent and identically distributed (i.i.d.) random variables will approximate a normal distribution, regardless of the original distribution of the variables.

Sample Mean Distribution: The CLT posits that the distribution of the sample mean will tend to a normal distribution as the sample size increases. This holds true even if the original population distribution is not normal.

Conditions: The theorem requires that the sample size be sufficiently large (usually $n \geq 30$), the samples are independent, and the population has a finite variance.



Sample size v number of samples

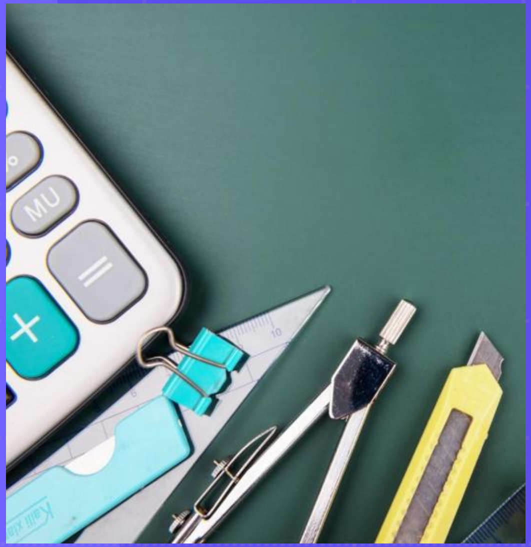
Sample size is to how many people we are going to ask our question.

Number of samples is how many data points we will achieve by conducting our survey.

Example:

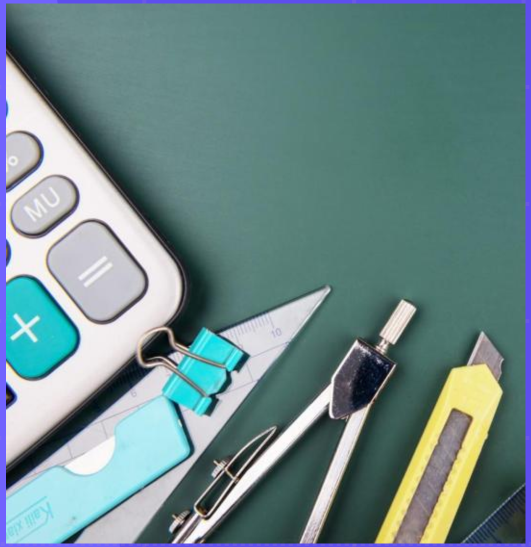
Everyone in this room (lets assume 100) goes outside, goes for a walk into Joondalup and everyone asks 50 people each a question. We all calculate our sample mean and report back.

Our sample size is 50, the number of samples we have conducted is 100.



Sample size v number of samples

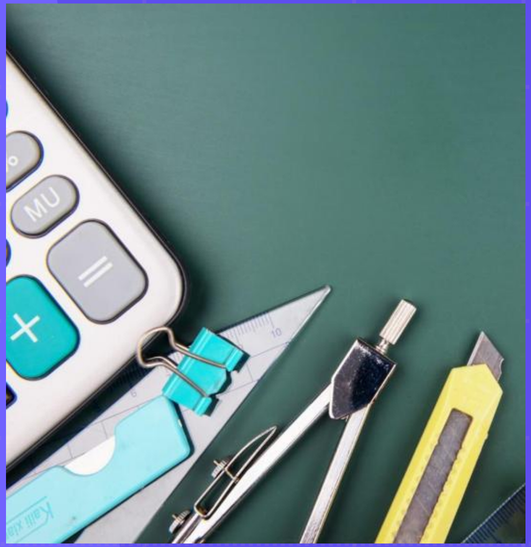
Usually as a researcher, you won't be able to conduct 100 samples and so usually your sample size is $n \geq 30$, but the number of samples we conduct is limited.



Central limit theorem (CLT)

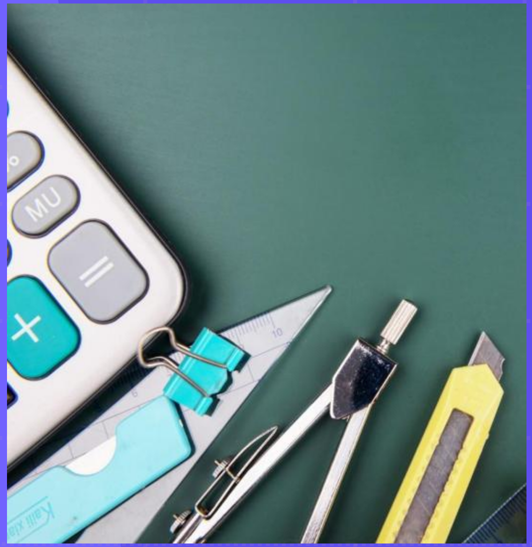
As the sample size of our samples increases, the distribution of our sample means approaches a **normal distribution**. This is due to the central limit theorem. This is irrespective of what the distribution of our samples were.

- If our population or samples are <u>normally distributed</u>, then we can assume that the sample means will be normally distributed regardless of the size of the samples.
- If our population or samples are <u>not normally distributed</u>, we can only expect our sample means to be normally distributed when the size of the samples are larger than or equal to 30, that is, $n \geq 30$. (More on this on a following slide)
- The mean of sample means, $\bar{x}$, approaches the population mean, μ.
- The standard deviation of the sample means approaches $\frac{\sigma}{\sqrt{n}}$, where σ is the standard deviation of the population σ.



Central limit theorem (CLT)

Distribution	What it describes	Its shape
Population distribution	Every individual value in the whole population	Whatever it is: skewed, flat, bimodal
One sample	The n values you actually collected	Roughly mirrors the population shape
Sampling distribution of the mean	The means of all possible samples of size n	Approximately normal once n is large

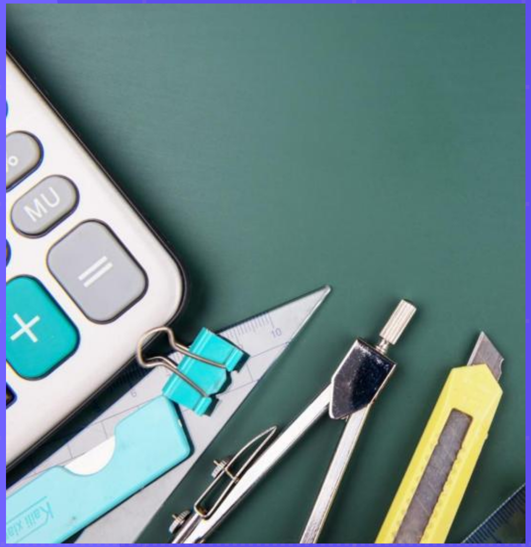


So how big does the sample need to be?

The phrase "n of at least 30" appears so often that many students treat it as a law. It is a convention. The honest answer is that the sample size you need depends on how far the population is from being symmetrical.

Population shape	**Roughly how large n needs to be**	**Why**
Already normal	Any n, even n = 1	The sample mean is exactly normal for every size
Symmetric, not normal	Small, often 10 to 15	No skew to cancel out, so it settles fast
Moderately skewed	About 30	The standard rule of thumb fits this case
Strongly skewed or heavy-tailed	50, 100, or more	Long tails take many values to balance

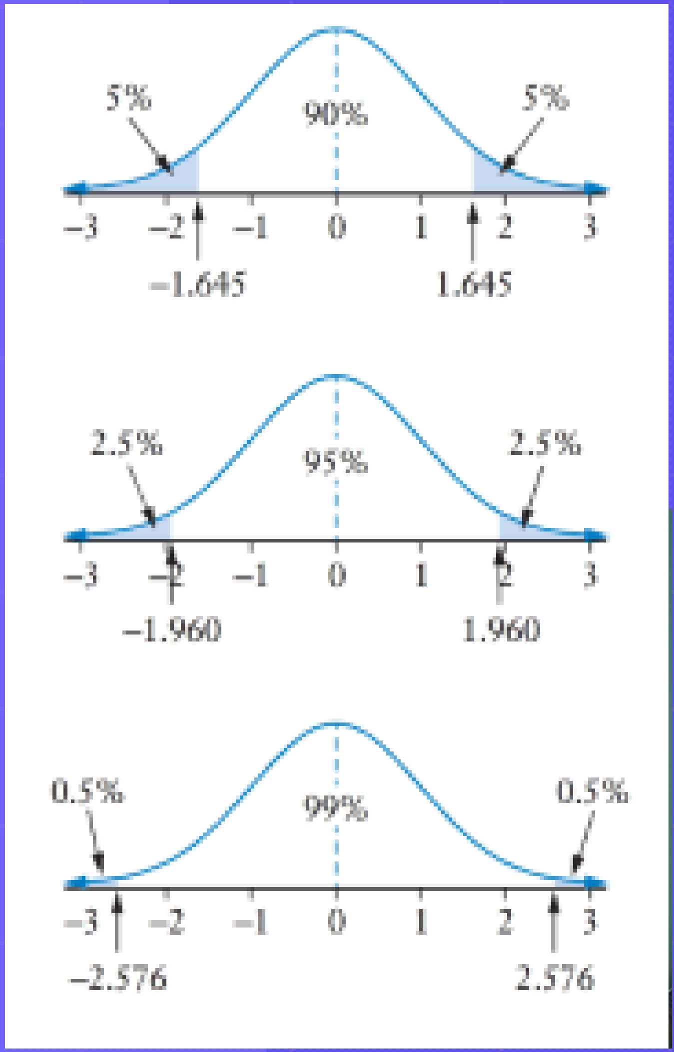
"The Central Limit Theorem says every distribution becomes normal at n = 30" is false on two counts. The population never becomes normal, and 30 is not a fixed cutoff. A strongly skewed distribution can still produce skewed sample means at n = 30 and may need several hundred observations before the bell shape holds.



The standard normal distribution

You will recall from Methods that the standard normal distribution is a normal distribution with a mean of 0 and standard deviation of 1. It is also associated with z-scores, and we commonly look at the confidence levels of 90%, 95% and 99%.

- 90% of the distribution will lie within 1.645 standard deviations of the mean.
- 95% of the distribution will lie within 1.960 standard deviations of the mean.
- 99% of the population will lie within 2.576 standard deviations of the mean.



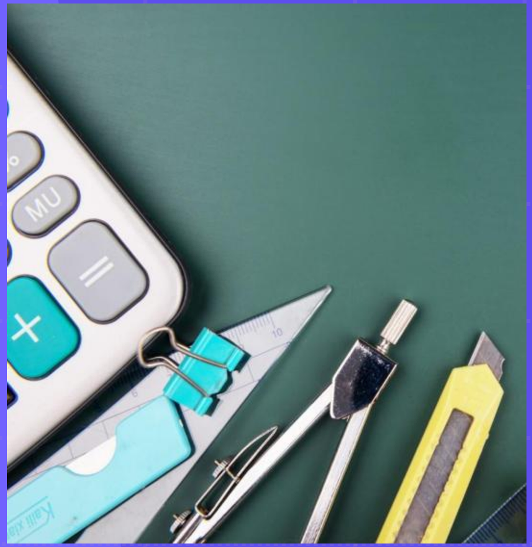
The standard normal distribution

As we have discovered from the central limit theorem, the sample means belong to a normal distribution (given $n \geq 30$).

If a given sample mean lies further than 1.645 standard deviations of this distribution we say there is 'a significant difference at the 10% level'.

This might indicate that this particular sample mean is unusual in some way or may contain some error. Thus further sampling may be required.

This is therefore saying that this particular sample mean lies in one of the tails or extremes of the distribution, i.e. where only 5% of the sample means should lie and may not be a true representation of the population.



The standard normal distribution - revision

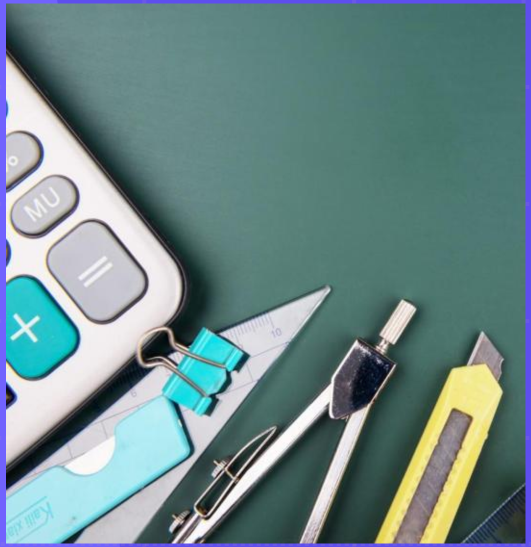
Key Features of the Standard Normal Distribution

Mean and Standard Deviation: The mean is 0, and the standard deviation is 1, making it a standardized form of the normal distribution.

Symmetry: The curve is symmetric around the mean, with probabilities decreasing symmetrically as values move away from the centre.

Total Area: The total area under the curve equals 1, representing the total probability. This is true anytime you are looking at probability, since a probability of 1 is every possible outcome.

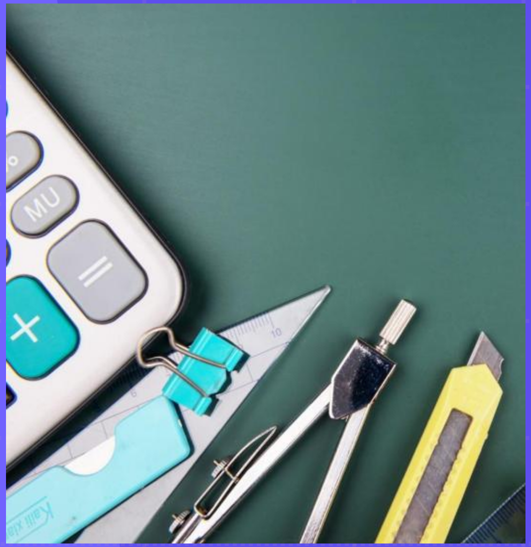
z-Scores: Any value from a normal distribution can be transformed into the standard normal distribution using the formula $z = \frac{(X-\mu)}{\sigma}$, where z is the z-score.



Confidence intervals (interval estimates)

A confidence interval is a range of values which is likely to contain (in repeated sampling) the true value of an unknown statistical parameter, such as a population mean.

- is a range of values - given above and below the sample mean (remember CLT says the sample mean will approximate the population mean
- likely to contain - but does not have to contain!...... This is often a stumbling block for people, we can not be sure that the mean is contained within the confidence interval.
- (in repeated sampling) - one sample doesn't adequately describe the population, and we don't know if the one sample will describe the population.
- the true value - the true value is the population mean, and this is a <u>fixed value</u>, which we do not know. The sample mean we calculate is a <u>continuous random variable</u> and changes with each sample

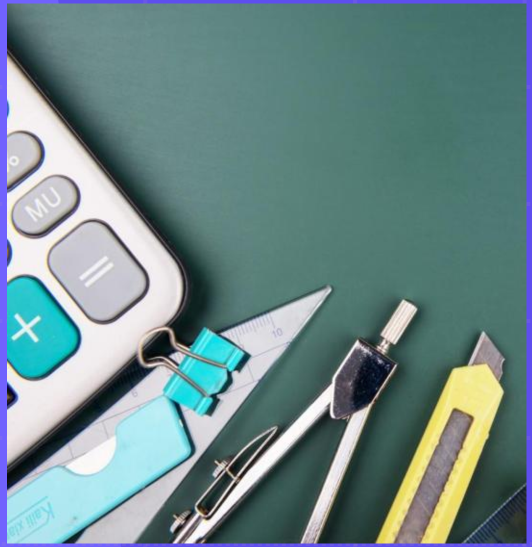


What does a confidence interval mean?

A 95% confidence level does not imply a 95% probability that the true parameter lies within a particular calculated interval, as we couldn't say that.

Remember the true mean is a fixed value, but the sample mean is a continuous random variable. Because the sample mean is random, the end points of the confidence interval are random as well.

However, we can say if the same sampling procedure were repeated 100 times from the same population, approximately 95 of the resulting intervals would be expected to contain the true population mean.



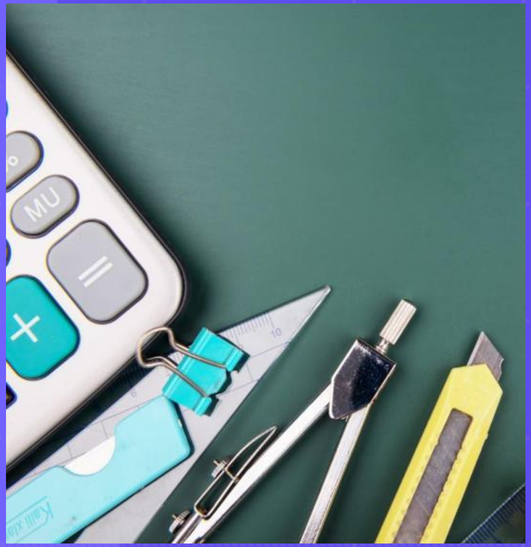
Rounding a confidence interval

We must pay attention to rounding in confidence intervals as rounding a calculated value can alter the actual confidence interval being calculated.

Let's assume you calculate a 95% confidence interval, with a sample standard deviation of 12, and a mean of 125, with 50 samples.

Doing the maths, you should get 121.6 < mean <128.4

Without knowing what the original data was (are only whole numbers suitable? Should we go to multiple decimal places?) we don't know exactly how to express our answer.

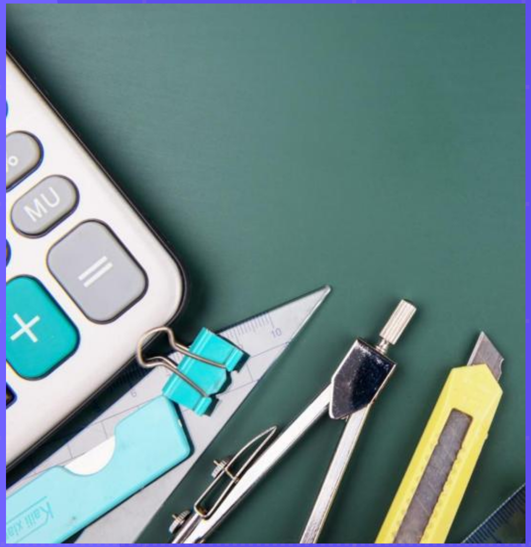


Rounding a confidence interval

If we were to round to the nearest whole number, that is from 121.6 < mean <128.4 to 122 < mean < 128

This has reduced our confidence interval as if we used a lower of 122 and upper of 128, we would get a value less than 0.95 for the area contained within that range on our normal distribution. So, the question then is, have we answered the question?

Be very aware that rounding can alter the question/answer/accuracy and hence we should be very cautious of 'rounding' in any fashion on these questions. If a question used the language of "at least a 95% confidence interval" then we could use this language to give us guidance on the appropriateness of rounding.



Determining the required sample size

Given that a confidence interval is created with the following formula

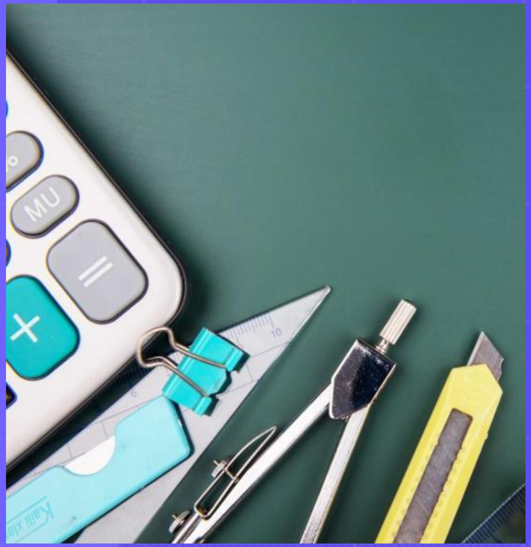
$$\bar{x} - k\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + k\frac{\sigma}{\sqrt{n}}$$

Hence if we know the population standard deviation (σ), our confidence interval's z score (k) desired and how many units with the mean (d) we want to be, we can use

$$k\frac{\sigma}{\sqrt{n}} = d$$

to determine the size sample we will need to take.

Be careful on rounding here too!



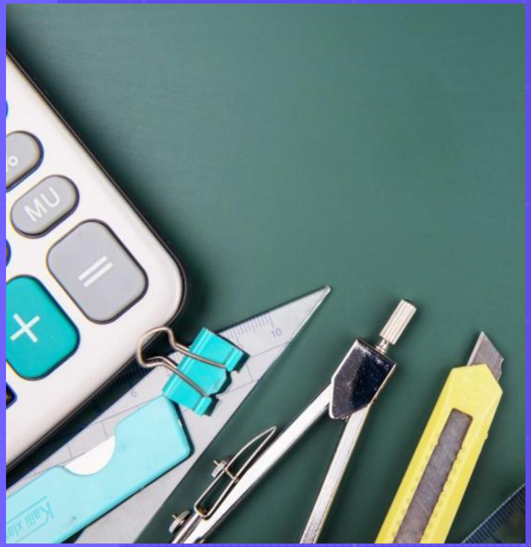
Formula sheet

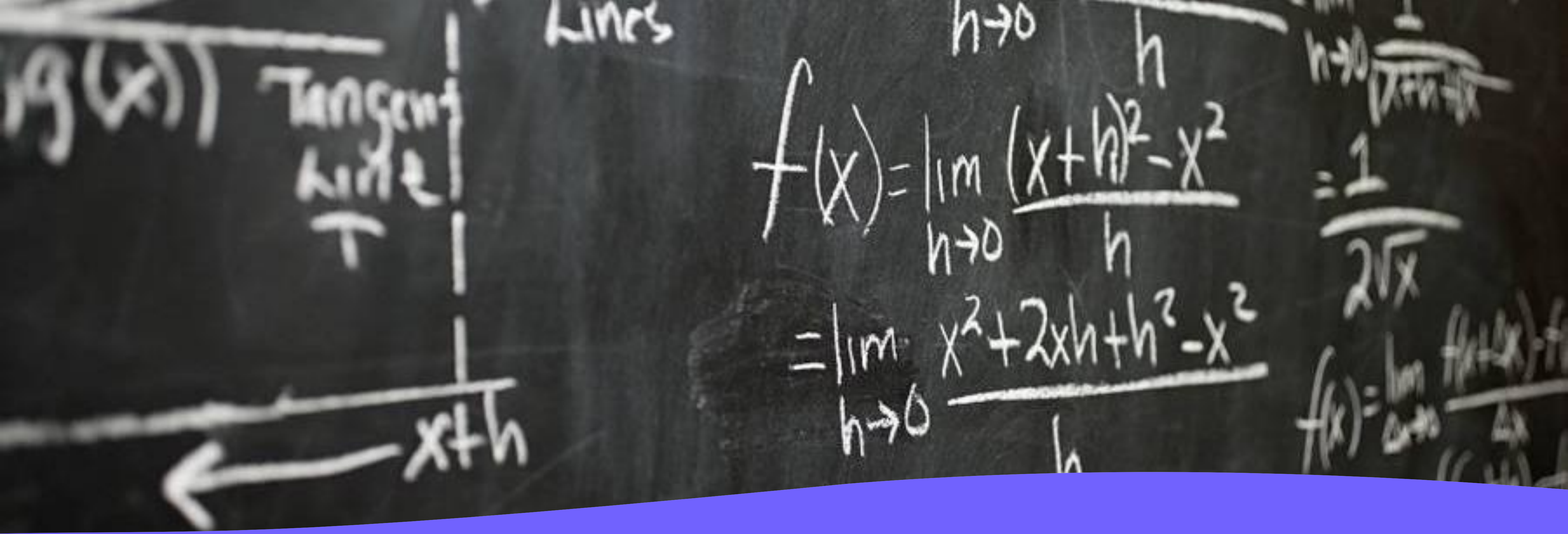
The only help we get on the formula sheet is this one!

Statistical inference

Confidence interval for the mean of the population	$\overline{X} - z\frac{s}{\sqrt{n}} \leq \mu \leq \overline{X} + z\frac{s}{\sqrt{n}}$
Sample size	$n = \left(\frac{z \times s}{d}\right)^2$

The variables use here are different to most text books.




Lines
Tangent
Line
x+h
$f(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$
$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$
$=\frac{1}{2\sqrt{x}}$

Questions?

WACE 2022 Question 12 (C)

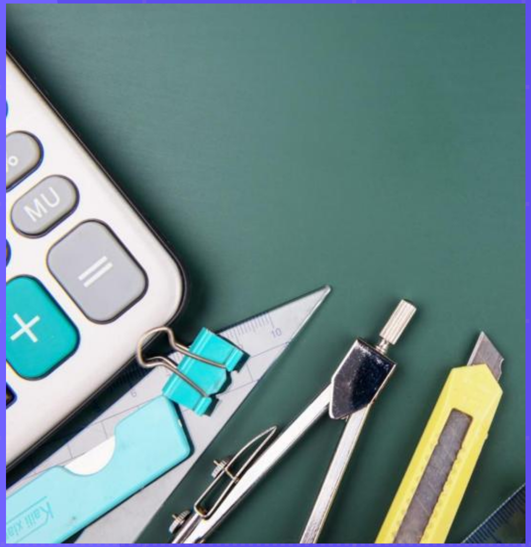
Question 12 **(9 marks)**

The inner diameter of a cylinder in a motor car engine is critical to its performance. Let μ mm denote the population mean cylinder diameter produced by a manufacturing process. A random sample, R_1 , of 100 cylinder diameters is taken and the standard deviation for this sample was found to be 1 mm.

Let $\bar{X}$ = the sample mean cylinder diameter for sample R_1 .

(a) State the distribution for $\bar{X}$ and its parameters. (3 marks)

(b) What is the probability that $\bar{X}$ differs from μ by more than 0.2 mm. Give your answer correct to 0.001. (2 marks)



WACE 2022 Question 12 (C)

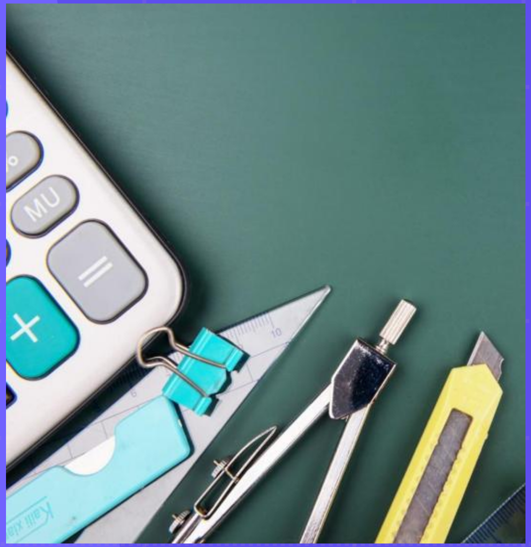
From random sample R_1, a 95% confidence interval for μ is formed.

(c) Calculate the width of this confidence interval, correct to 0.001. (2 marks)

Lilian, the production manager, wishes to decrease the width of the confidence interval. She suggests:

"We can form sample R_2 by using the data from sample R_1 and then combining this data with itself to form a sample with 200 observations. Using $n = 200$ will decrease the width of the confidence interval."

(d) State **two** major problems with using this idea. (2 marks)



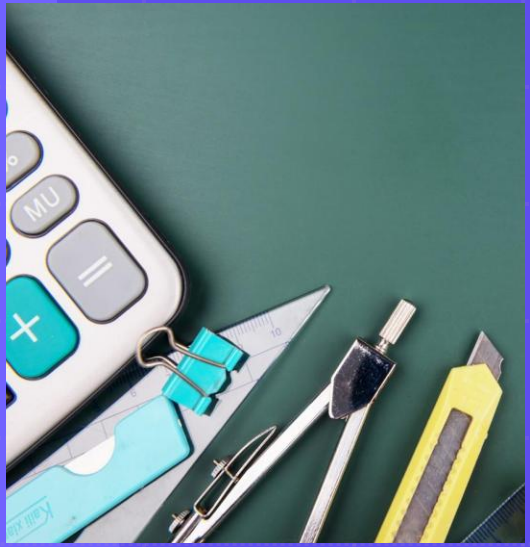
WACE 2023 Question 15 (C)

Question 15 **(9 marks)**

The WeLuvYas Bank extends personal loans to approved customers. A random sample of n personal loans is taken. A 99% confidence interval for the population mean loan μ (in thousands of dollars) based on this sample is $10.2 < \mu < 25.4$.

(a) What is the mean personal loan $\bar{x}$ for this sample? (2 marks)

(b) Calculate the standard deviation of the sample mean. (2 marks)



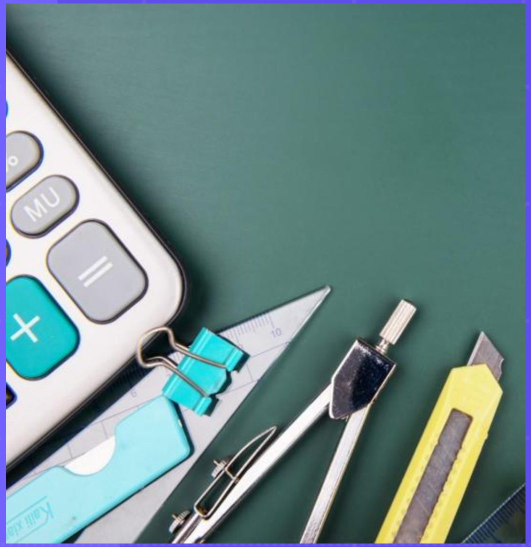
WACE 2023 Question 15 (C)

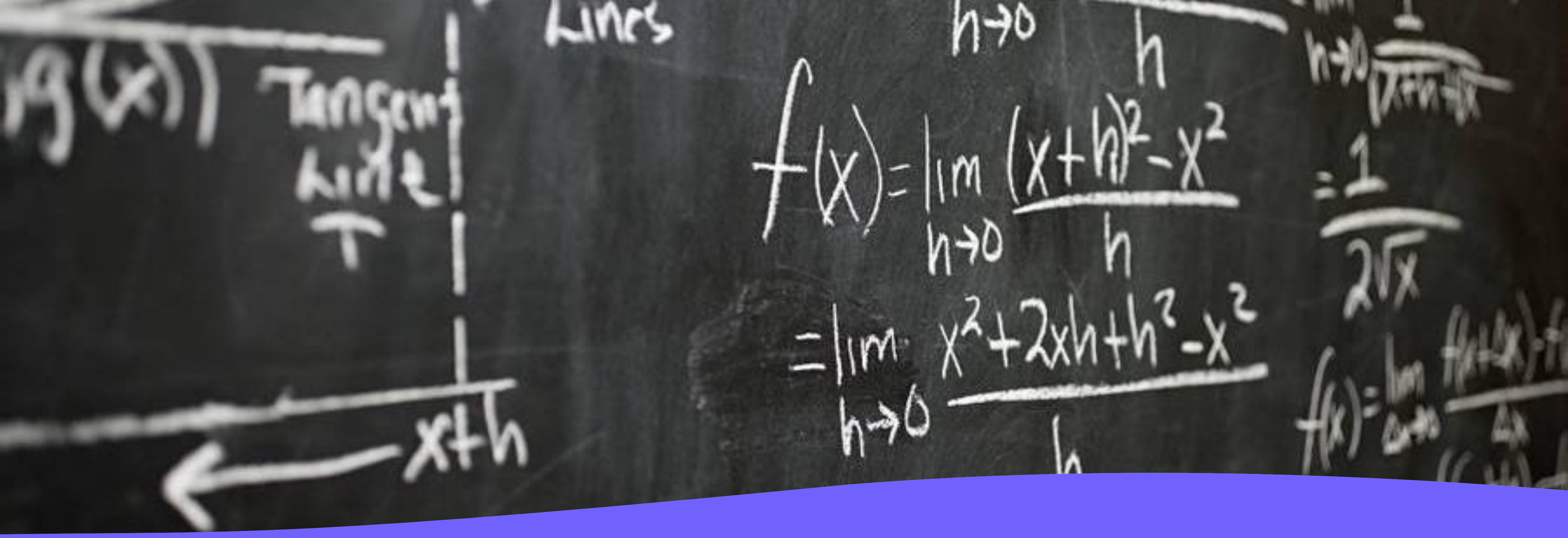
Ali exclaims excitedly 'everyone here at WeLuvYas is 99% certain that the true population mean is within the interval $10.2 < \mu < 25.4$'.

(c) State **two** reasons why Ali is not correct. (2 marks)

A data analyst discovers that the sample size was actually $2n$. In addition to this, the sample mean was actually $2000 more than that originally determined.

(d) Re-calculate the 99% confidence interval for the population mean on the basis of the updated information. (3 marks)





Formula Sheet

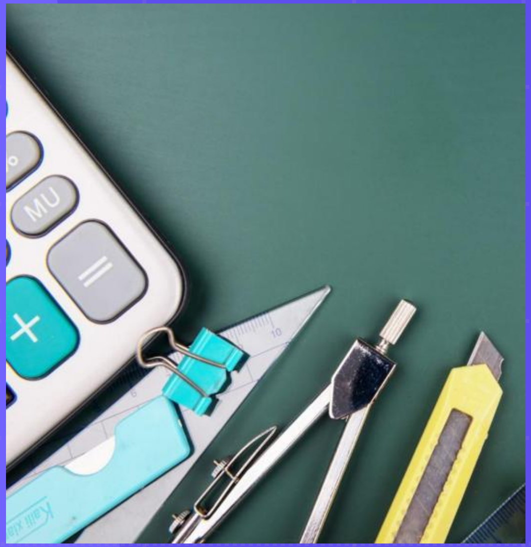
Personal Notes and Formula Sheet

Through out the presentation I have included snips of the Formula Sheet you will have access to in the exam (will be provided).

On your personal notes, do not waste space and time rewriting the Formula Sheet!

Should further equations, information, etc be needed it will be provided in the question

Note: Any additional formulas identified by the examination panel as necessary will be included in the body of the particular question.



Formula Sheet

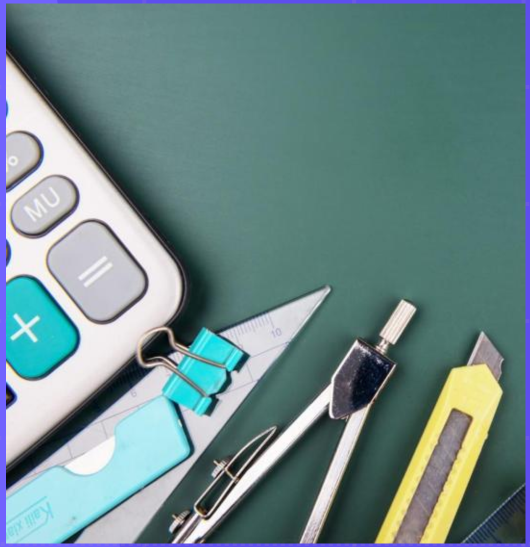
The following parts of your formula sheet have not appeared in this presentation.

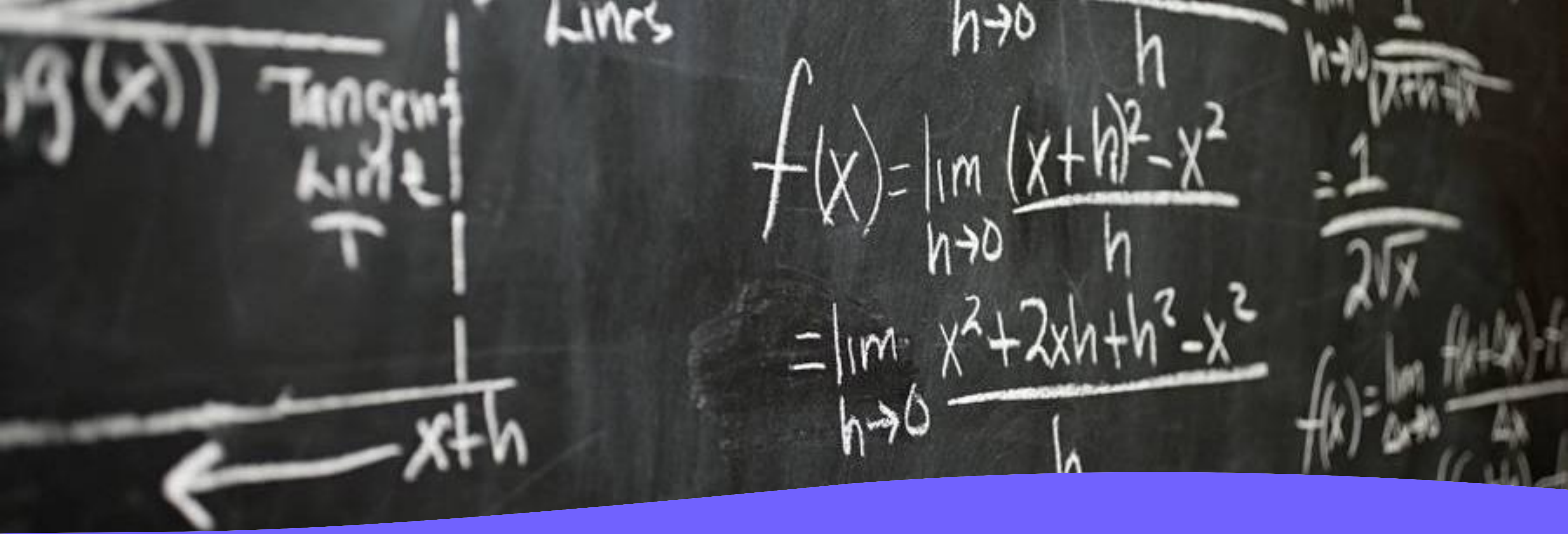
Mensuration might be helpful in determining area under a curve? Rotation around an axis? Maybe?

Mensuration

Parallelogram	$A = bh$
Triangle	$A = \frac{1}{2}bh$ or $A = \frac{1}{2}ab\sin C$
Trapezium	$A = \frac{1}{2}(a+b)h$
Circle	$A = \pi r^2$ and $C = 2\pi r = \pi d$

Prism	$V = Ah$, where A is the area of the cross section	
Pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base	
Cylinder	$V = \pi r^2 h$	$TSA = 2\pi rh + 2\pi r^2$
Cone	$V = \frac{1}{3}\pi r^2 h$	$TSA = \pi rs + \pi r^2$, where s is the slant height
Sphere	$V = \frac{4}{3}\pi r^3$	$TSA = 4\pi r^2$





Time to take a break!

10 minutes to stretch your legs, have a chat, use the restrooms!

WACE Exam – the details

Game time is…..

Monday 2nd November – 1 month away

	Monday	9 November	FST: Food Science and Technology *MAS: Mathematics Specialist	B

Start time is 9:20am, be there for 8:30am

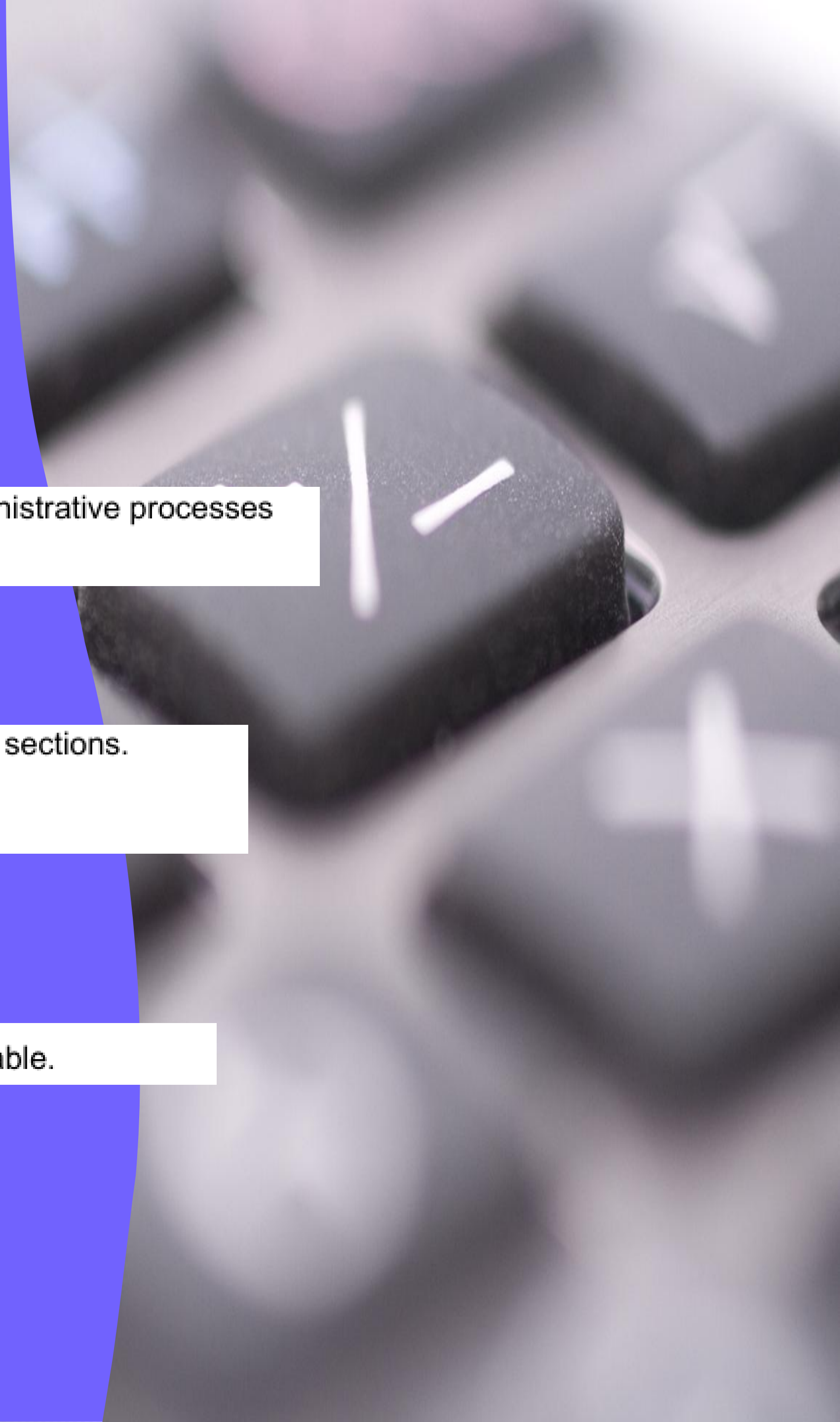
If you have an exam at a venue which isn't your school, make sure you know where it is!

WACE Exam – the details

- Candidates must arrive 30 minutes prior to the examination start time. This will allow time for seating, administrative processes and the reading of instructions.

- * Mathematics Applications, Mathematics Methods and Mathematics Specialist examinations consist of two sections.
  - Section one: Calculator-free has 5 minutes reading time and 50 minutes working time.
  - Section two: Calculator-assumed has 10 minutes reading time and 100 minutes working time.

- No allowance can be made for candidates who miss the examination session through misreading the timetable.



WACE Exam – Handwriting!

- Write neatly and legibly!
- Make sure your fours (4) look like a 4, sevens (7) are also pretty bad!
- This isn't English or Lit, there is no reward for writing more….

WACE Exam – Hints and tips

Recognise the go no where question!

You read the question,

You try something,

You try again,

You stare at it……..

5 minutes disappears!!!! Or maybe 10 minutes gone!!!! And maybe 1 mark gained…

WACE Exam – Hints and tips

Instead

🛑 STOP

➡️ Mark the question that you need to come back to it
➡️ Move on and get your marks elsewhere
➡️ Complete questions you can answer and build confidence
➡️ Return with fresh thinking if you have time

Why? Because 5 minutes spent stuck = marks lost elsewhere.

WACE Exam – Hints and tips

- 1 mark $\approx$ 1 minute
- Stuck? Move on
- Spending some time on a question? Getting somewhere, stick with it; not getting anywhere, move on
- 1 mark is for one key skill shown, it isn't for how many lines of working you do
- Some answers can be two or three lines for 5 marks! But sometimes you'll need multiple lines to get 1 or 2 marks.

WACE Exam – Hints and tips

Reading time

- Decide the 'easy', 'medium', 'hard' questions
- Spot your timewasters – "I have no idea!"
- Plan your approach
- Use your formula sheet/notes to plan some questions
- The wordier questions, have a read/reread to ensure you're understanding them correctly.

WACE Exam – Hints and tips

Working time

- Write neatly!!!!
- Don't start writing half way across the page!
- One mark is one skill, 10 lines isn't 10 marks
- Show your logical thought through your working
- Make your final answers clear
- If you mess a question up pretty bad, ask for another booklet!

WACE Exam – Hints and tips

Classpad skills

- Don't write "I did it on my Classpad"!!
- Show the substitution into the equation → then write the answer from Classpad ☺
- If your Classpad freezes/jams up/gives up on a question, you did something wrong in the question!
- If you haven't changed your batteries…. Do it now!

WACE Exam – big questions?

What can I do between now and the exam to make my marks more predictable?

What can I do between now and the exam to push my marks higher?

What can I do between now and the exam to build my understanding?

WACE Exam – Think about?

What part of the exam worries you most?

What do you already feel confident about?

What has helped from today's session?

Maybe another question you have?

WACE Exam – Know yourself

What are your strengths?

Examples:

- Content knowledge
- Visualisation of question
- Mental maths calculations
- Classpad use
- Time management
- Reading and deciphering questions
- Remembering and applying terminology

WACE Exam – Know yourself

What are your weaknesses?

Examples:

- Panic (why? What are your triggers? What can you do before the exam to minimise?)
- Running out of time – spending too long on some questions?
- Misreading questions –understanding? Maybe rushing? Lack of content knowledge?

WACE Exam – Know yourself

What are your weaknesses?

Examples:

- Not showing working!!!! This is often a big one for some people
- Writing too much – yes! for some
- Writing too little –yes! for some
- Getting stuck (and not moving on?)

WACE Exam – Know yourself

Before the exam, I will focus on improving ….

In exams my biggest mistake to avoid is….

One strategy I need to use more in the exam is…

Make your responses specific, measurable and implementable.
"Study more" isn't a good response.

WACE Exam – the markers?

Put yourself in the place of the marker

- Is marking papers easy… well mostly yes, but not always
- Following the thought pattern of students is the hardest part
- Big jumps in thinking, do they really know what they're doing? Or is this rote learnt?
- When a marker is lacking confidence in a student, marks are unlikely
- We can only mark what we see, not what you're thinking
- Neatness and logical thought process are always the winner!

WACE Exam – Study/revision

Don't rely only on:

📖 Re-reading textbook chapters

📖 Looking over notes/examples

🖍️ Highlighting/underlining etc

👀 Watching videos

WACE Exam – Study/revision

What does effective revision look like then for Mathematics?

To revise Mathematics, we need to triangulate three key aspects

1. Building content knowledge – syllabus, book, knowledge
2. Understand what the questions are asking and testing
3. Be able to complete the mathematics to show our content knowledge

WACE Exam – Study/revision

Okay, great how do we do that then?

WACE Exam – Study/revision

1. Building content knowledge – syllabus, book, knowledge

For each topic (6 in our course), ask:

WHAT? What do I need to revise? Check syllabus!

HOW? What method will I use? More questions, more understanding, more what?

WHEN? When will I do it?

HOW LONG? How much time will I realistically spend?

WACE Exam – Study/revision

2. Understand what the questions are asking and testing

To do well here, we need:

- To know our syllabus
- To recognise the types of questions that come up
- Be able to show the required skills that get rewarded with marks

WACE Exam – Study/revision

3. Be able to complete the mathematics to show our content knowledge

If the first two are being done well, then the third step should naturally follow but we must pay attention to

- Computation errors
- Paying attention to units, scales, sizes, orders of magnitude
- Showing what you are doing and not jumping steps
- Don't write 'I did it on my Classpad' no no no!

WACE Exam – Study/revision

Some other points you need to consider in your revision are

☕ Brain breaks

🏃 Movement

📉 Downtime

💧 Water

😴 Sleep

📵 Focused study time

🏢 Work/sport/family etc

WACE Exam – Study/revision

REVIEW IT

Refresh the knowledge (syllabus, text book, notes, videos, past questions etc)

RETRIEVE IT

Close the notes and recall it (think through the content, the skills, the questions)

APPLY IT

Complete a question (find a past WACE question from the topic you're revising)

MARK IT

Identify what earned/lost you marks.

WACE Exam – Study/revision

FIX IT

Learn from the mistakes

REVISE IT

Relook at the aspect(s) that weren't strong for you

RETRY IT

Flag the question to attempt it again later

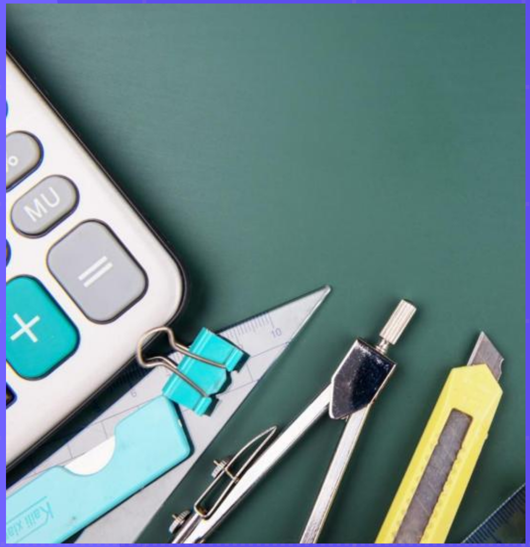
WACE Examiner's Reports

These are a treasure trove of information! You must consult them and learn from other's mistakes! (you don't have enough time to make all the mistakes for yourself!)

There are two versions – public and 'for teachers'

2025 Examination report (333.7 KB) Last updated: 19 Mar 2026 2:28pm

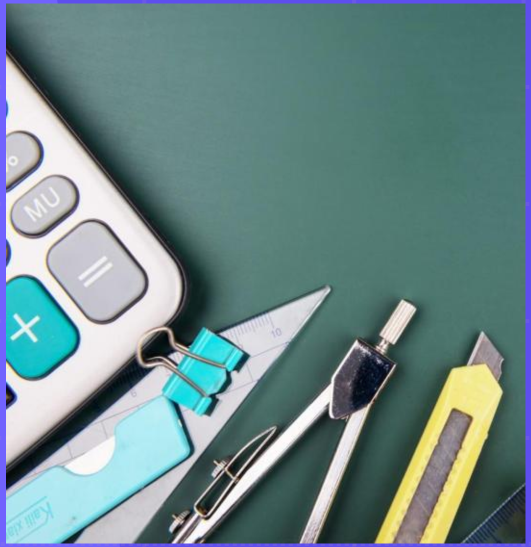
2025 Summary examination report for candidates (272.8 KB) Last updated: 19 Mar 2026 2:28pm



You must read the full version, the summary is somewhat useless!

WACE Examiner’s Reports

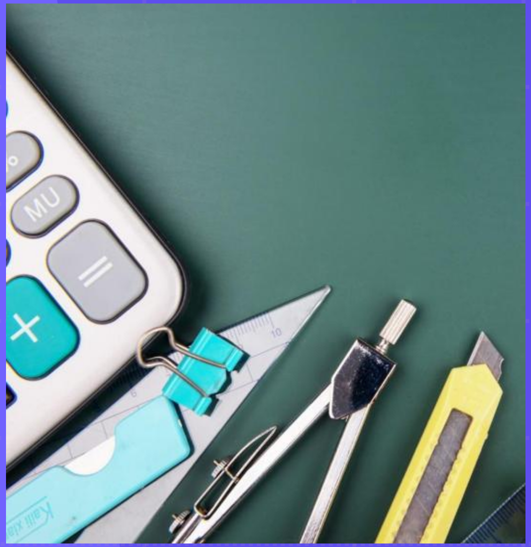
2025-ATAR-course-examination-report-MAS.PDF



2026 WACE exam front covers

2026-MAS-Examination-Calculator-Free-Front-Cover.PDF

2026-MAS-Examination-Calculator-Assumed-Front-Cover.PDF



WACE Questions

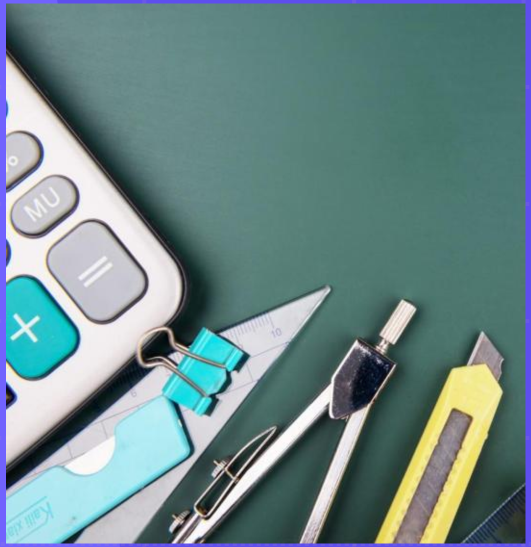
Now let’s use what we’re revised and learnt to hit some WACE questions

Ahead are 6 questions, one from each topic

WACE 2023 Question 6 (U3T1)

Question 6 **(5 marks)**

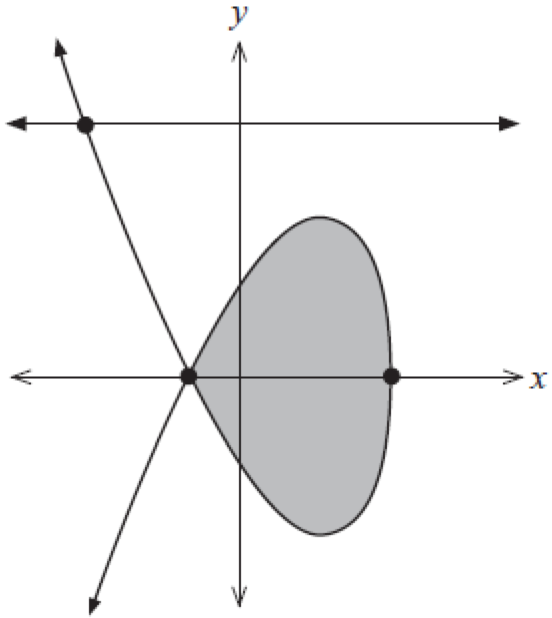
Solve the complex equation $z^4 = 2 - 2\sqrt{3}i$ giving solutions in the form $r\text{cis}\theta$ where $-\pi < \theta \leq \pi$.

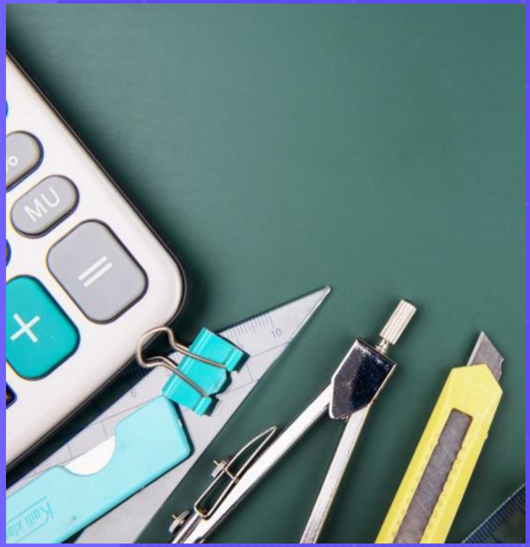


WACE 2022 Question 13 (U3T2)

Question 13 **(8 marks)**

The equation $x^3 - x^2 - 5x = 3 - y^2$ implicitly defines the curve shown below. The line $y = \sqrt{24}$ intersects this curve as shown below.



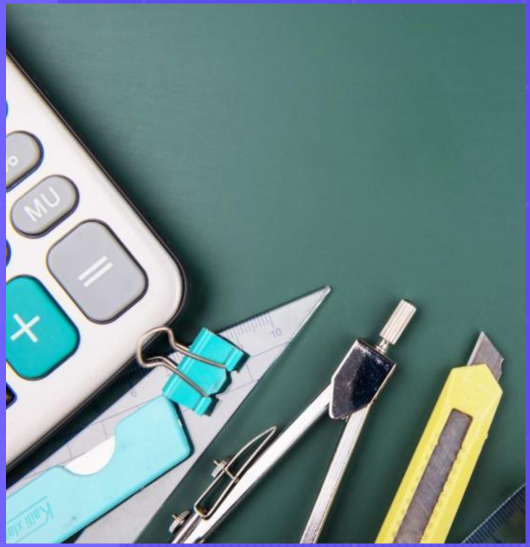


WACE 2022 Question 13 (U3T2)

It can be shown that the equation $x^3 - x^2 - 5x + 21 = 0$ will determine the intersection between the line $y = \sqrt{24}$ and the implicitly defined curve.

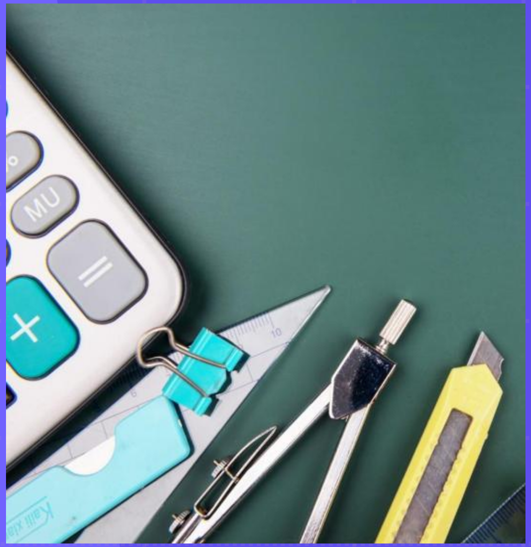
(a) Explain, with reference to the graph above, why we know that there is one real and two complex solutions (a conjugate pair) to this cubic equation. (2 marks)

(b) Determine the **two** exact complex solutions to the equation $x^3 - x^2 - 5x + 21 = 0$. (2 marks)



WACE 2022 Question 13 (U3T2)

(c) Calculate the area of the shaded region, correct to 0.001 square units. (4 marks)



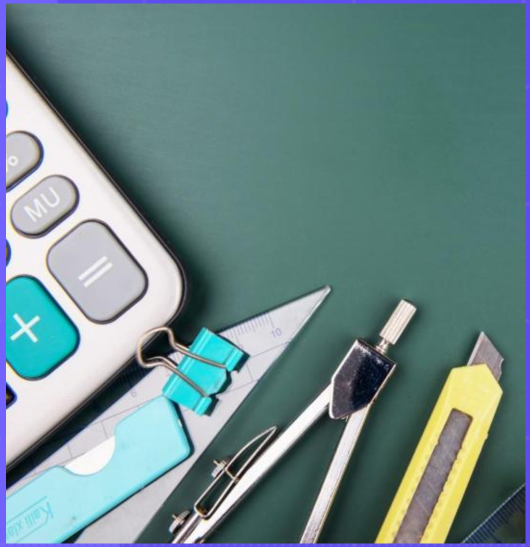
WACE 2022 Question 14 (U3T3)

Question 14 **(11 marks)**

Plane P_1 has Cartesian equation: $z = 2x + y + 4$.

Line L has equation given by: $\underset{\sim}{r} = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 2\lambda \end{pmatrix}$.

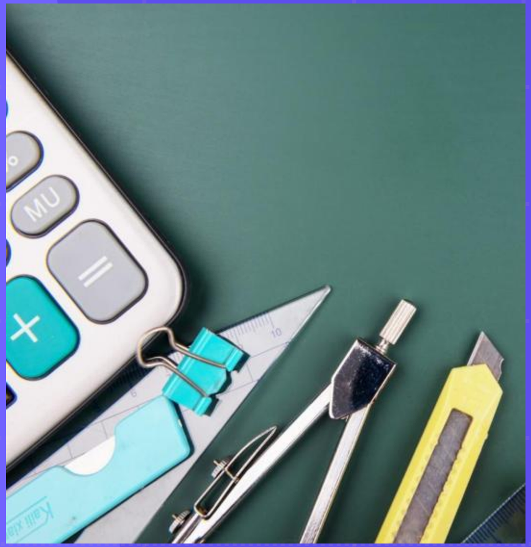
(a) Determine a vector that is perpendicular to plane P_1. (2 marks)



WACE 2022 Question 14 (U3T3)

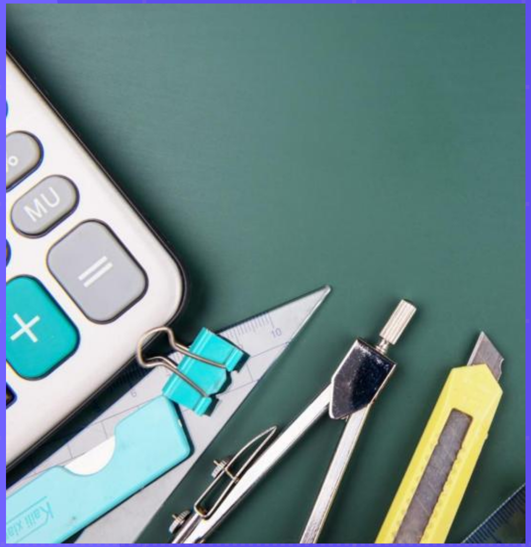
(b) Write the equation for plane P_1 in vector form. (2 marks)

(c) Determine the acute angle, correct to the nearest degree, between plane P_1 and line L. (3 marks)



WACE 2022 Question 14 (U3T3)

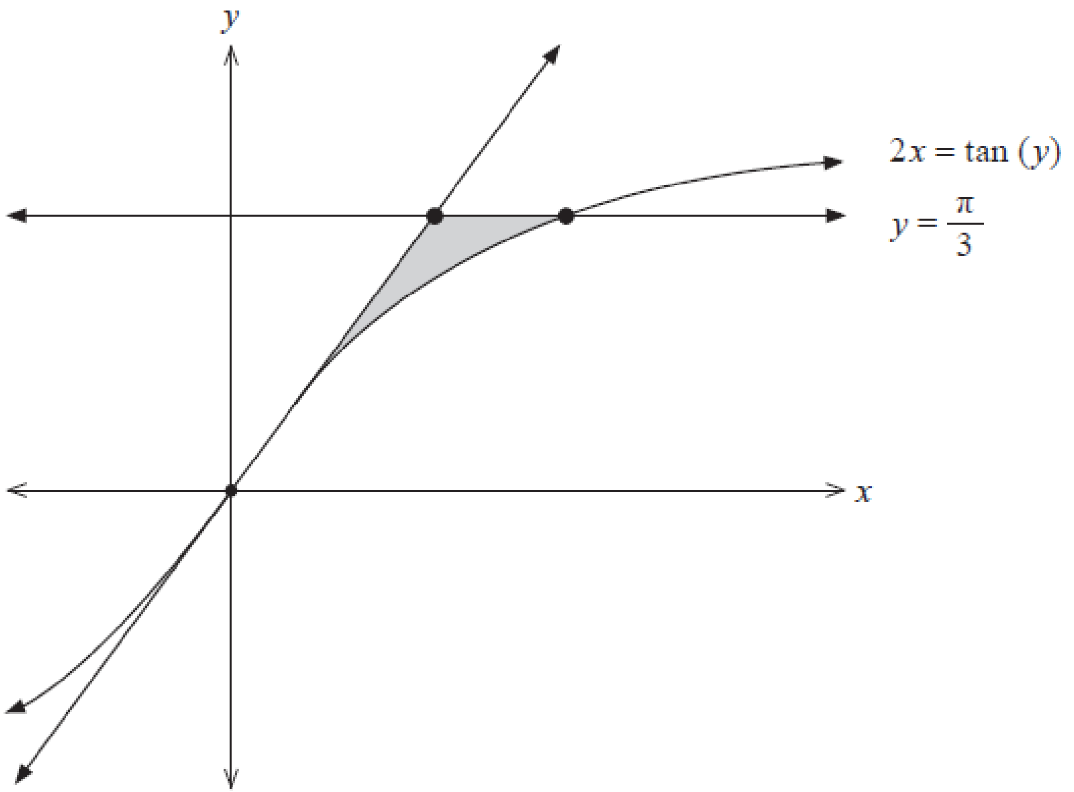
(d) Obtain the Cartesian equation of the plane P_2 that contains the line L and is perpendicular to plane P_1. (4 marks)

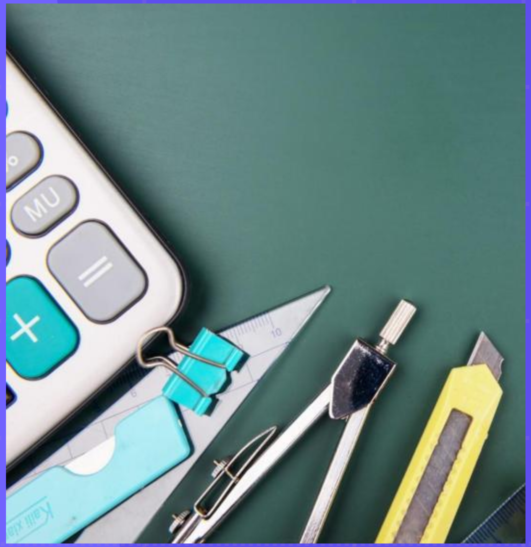


WACE 2022 Question 7 (U4T1)

Question 7 **(8 marks)**

The graph of $2x = \tan(y)$ is shown along with the tangent at $x = 0$. The horizontal line $y = \frac{\pi}{3}$ is also shown.





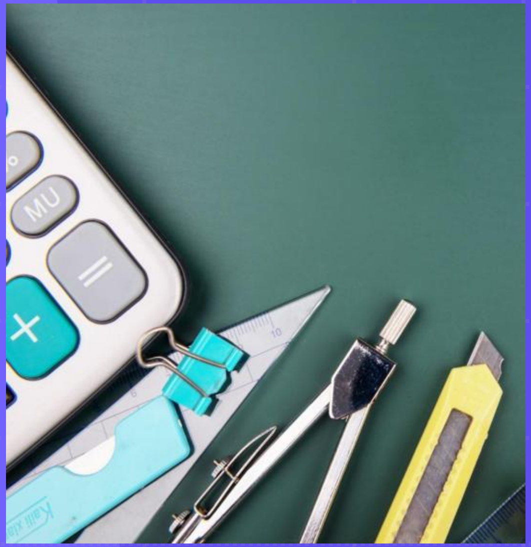
WACE 2022 Question 7 (U4T1)

(a) Using implicit differentiation, determine the equation of the tangent drawn at $x = 0$.
(3 marks)

The shaded region is bounded by the curve $2x = \tan(y)$, the tangent drawn and $y = \frac{\pi}{3}$.

(b) Write the expression for the area of the shaded region. (2 marks)

(c) Evaluate this area exactly. (3 marks)

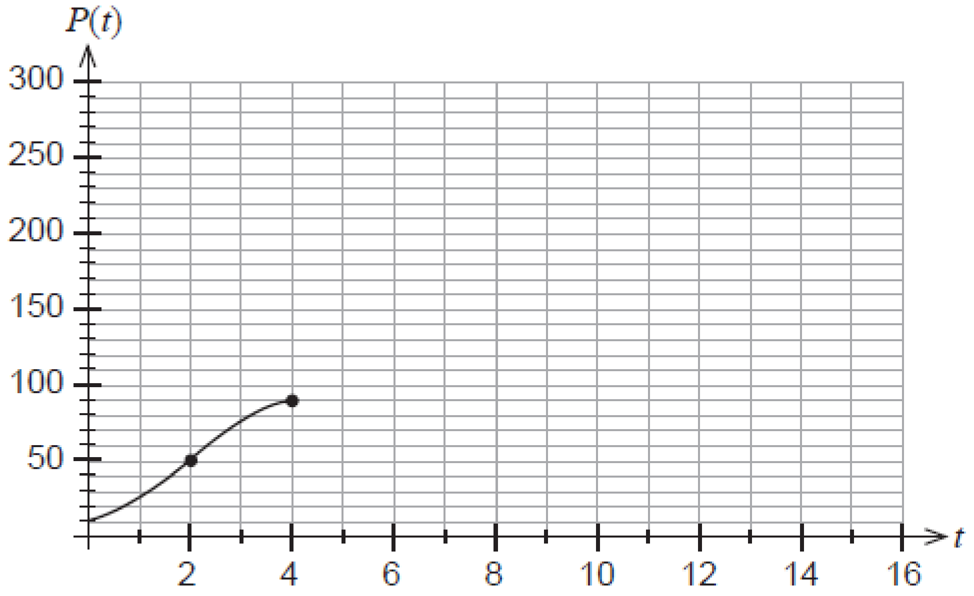


WACE 2022 Question 16 (U4T2)

Question 16 **(4 marks)**

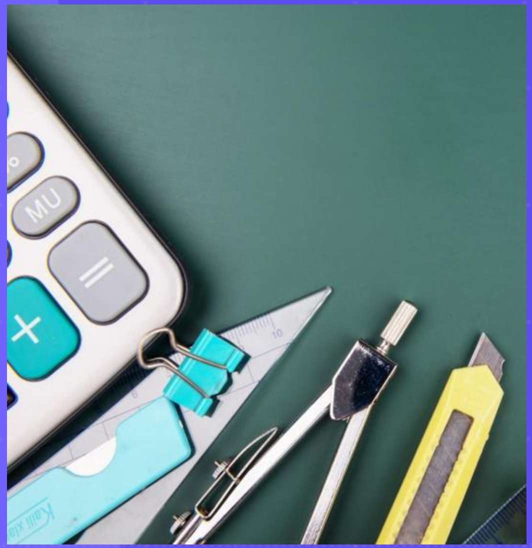
An ant colony population P at time t days grows at a rate given by the equation

$\frac{dP}{dt} = 0.01P(100 - P)$, where $0 \le t \le 4$. The graph of this population is shown below.



(a) For $0 \le t \le 4$, using the growth rate equation explain the variation of the population.

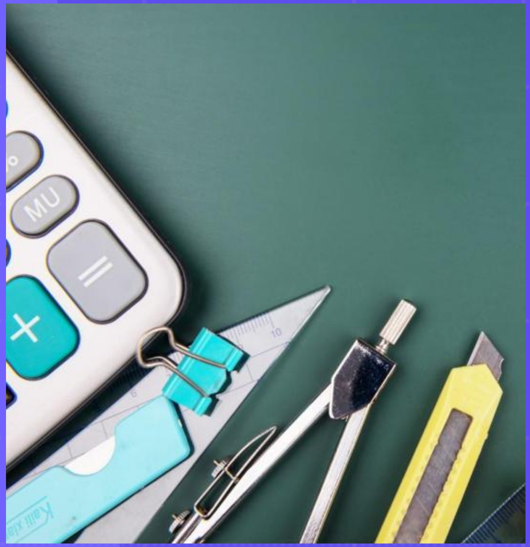
(2 marks)



WACE 2022 Question 16 (U4T2)

At the end of the fourth day, the environment for the ant colony improves dramatically so that its limiting population is increased to 300.

(b) Sketch, on the axes above, the expected variation of the population for $t > 4$ days, using the increased limiting population value. (2 marks)



WACE 2023 Question 2 (U4T3)

Question 13 **(11 marks)**

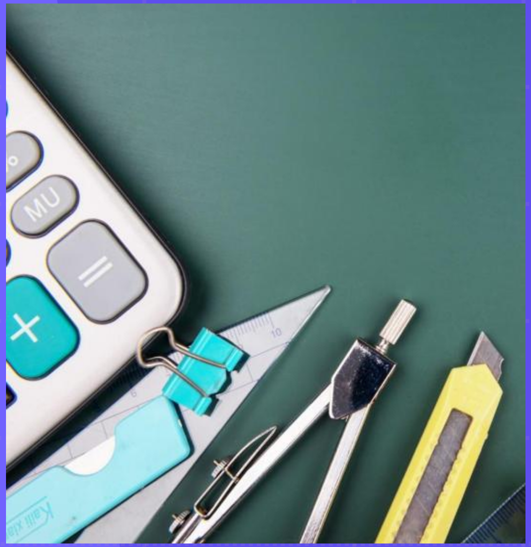
A factory produces boxes of breakfast cereal with a labelled weight of 1.00 kg.

Let μ denote the population mean and σ denote the population standard deviation of the weights of the boxes. The factory sets the packaging process to a mean weight $\mu = 1.01$ kg with a standard deviation $\sigma = 0.05$ kg.

To maintain quality, a random sample of 400 boxes is taken each day and weighed. Let $\overline{X}$ denote the sample mean weight.

(a) State the distribution for $\overline{X}$ and its parameters. (3 marks)

(b) Determine the probability that the sample mean is more than 5 g above the labelled weight. (2 marks)



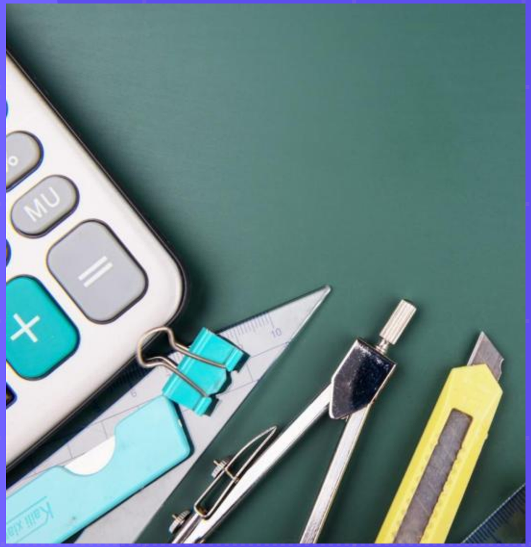
WACE 2023 Question 2 (U4T3)

The sample mean on a particular day is $\bar{x} = 1.05$ kg, while the sample standard deviation is $s = 50$ g.

(c) Determine a 95% confidence interval, correct to 0.001 kg, for the population mean weight based on this sample. (2 marks)

Anja, a quality control officer, wants a 95% confidence interval based on a sample size of 100 with a width of no more than 0.1 kg.

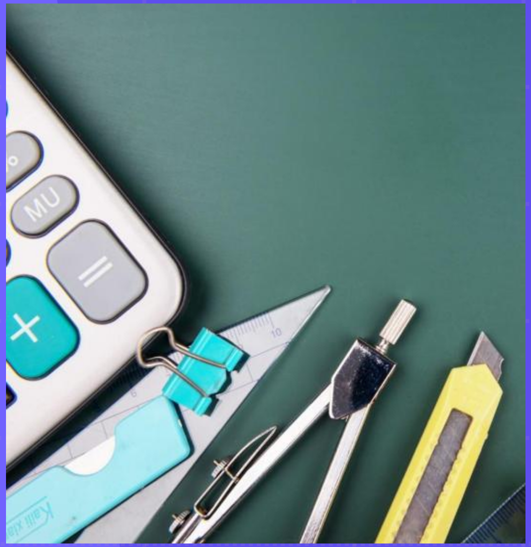
(d) What is the maximum standard deviation for this confidence interval? (2 marks)



WACE 2023 Question 2 (U4T3)

Over the next 50 days, Ben, who is a data collection agent, takes random samples of size 100 each day and a 95% confidence interval is calculated for each sample. Ten of these 50 intervals (20% of the intervals) have a lower bound that is less than 1.00 kg. Ben claims that this indicates that the mean weight of the packaging is set too low.

(e) Is Ben correct? Justify your response. (2 marks)



What’s next?

Well… you’ll all have to go home… I guess?
You can’t stay here!

Recognise where you are on the learning journey and be realistic about what you can achieve in the next four weeks.

What’s next?

Revise the things you know are going to feature in the exam, ignore the dot points about ‘develop’ or ‘investigate’

Past exam questions are one of the best vehicles to become more familiar with what is required and what is likely to reappear this year!

Finally.....

Thank you

For coming along
For being present here today
For furthering your mathematics
For challenging yourself to choose the harder maths

The way to get started is to quit talking and begin doing.

Walt Disney

Don’t aim for perfection, aim for progress.

Unknown

The world is a very dumb place, you are the intelligent ones, go forth and spread your intelligence!

Chris Simpson '26